UNIT-I
MATHEMATICAL MODELLING OF CONTROL SYSTEMS

Topics: Concepts of Control Systems- Open Loop and Closed Loop control systems. Mathematical modeling –Transfer function, Modeling of electrical systems, mechanical systems, Electrical analogy of mechanical systems. Block diagram representation of systems - Block diagram algebra. Signal flow graph – reduction using Mason's gain formula. Feedback Control System Characteristics- Sensitivity of Control Systems to Parameter Variations, Disturbance Signals in a Feedback Control System.

INTRODUCTION:
LAPLACE TRANSFORMATION OF COMMON FUNCTIONS

Function	Laplace Transform
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$

EXAMPLE-1: Find the Laplace transform of f(t) =1
SOL:

$$f(t) = 1$$

$$L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty 1 \cdot e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

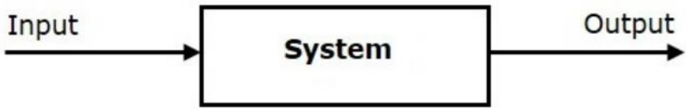
$$= -\frac{1}{s}\left[e^{\infty} - e^{0}\right]$$

$$= -\frac{1}{s}(0-1)$$

$$= \frac{1}{s}$$

CONCEPTS OF CONTROL SYSTEMS

System : *A system is a combination or an arrangement of different physical components which act together as an entire unit to achieve certain objective.*



Control system : *To control means to regulate, to direct or to command. Hence a control system is an arrangement of different physical elements connected in such a manner so as to regulate, direct or command itself or some other system.*

Plant : *The portion of a system which is to be controlled or regulated is called the* ***plant or the Process.***

Controller : *The element of the system itself or external to the system which controls the plant or the process is called* ***controller****.*

Input : *It is an applied signal or an excitation signal applied to a control system from an external energy source in order to produce a specified output.*

Output : *It is the particular signal of interest or the actual response obtained from a control system when input is applied to it.*

Disturbances : Disturbance is a signal which tends to adversely affect the value of the output of a system. If such a disturbance is generated within the system itself, it is called an **internal disturbance**. The disturbance generated outside the system acting as an extra input to the system in addition to its normal input, affecting the output adversely is called an **external disturbance**.

CLASSIFICATION OF CONTROL SYSTEMS

1) **Natural control systems :** The biological systems, systems inside human being are of natural type.

   Example 1 : The perspiration system inside the human being is a good example of natural control system. This system activates the secretion glands, secreating sweat and regulates the temperature of human body.

2) **Manmade control systems :** The various systems, we are using in our day to day life are designed and manufactured by human beings. Such systems like vehicles, switches, various controllers etc. are called manmade control systems.

   Example 2 : An automobile system with gears, accelerator, braking system is a good example of manmade control system.

3) **Combinational control systems :** Combinational control system is one, having combination of natural and manmade together i.e. driver driving a vehicle. In such system, for successful operation of the system, it is necessary that natural systems of driver alongwith systems in vehicles which are manmade must be active.

4) **Time varying and time - invariant systems : Time varying control systems are those in which parameters of the systems are varying with time. It is not dependent on whether input and output are functions of time or not.** For example, space vehicle whose mass decreases with time, as it leaves earth. The mass is a parameter of space vehicle system. Similarly in case of a rocket, aerodynamic damping can change with time as the air density changes with the altitude. As against this **if even though the inputs and outputs are functions of time but the parameters of system are independent of time, which are not varying with time and are constants, then system is said to be time invariant system.** Different electrical networks consisting of the elements as resistances, inductances and capacitances are time invariant systems as the values of the elements of such system are constant and not the functions of time.

5) **Linear and nonlinear systems : A control system is said to be linear** if it satisfies following properties.

a) The **principle of superposition** is applicable to the system. This means the response to several inputs can be obtained by considering one input at a time and then algebraically adding the individual results.

   Mathematically principle of superposition is expressed by two properties,

   i) Additive property which says that for x and y belonging to the domain of the function f then we have,

   $$f(x + y) = f(x) + f(y)$$

   ii) Homogeneous property which says that for any x belonging the domain of the function f and for any scalar constant α we have,

$$f(\alpha x) = \alpha f(x)$$

b) The differential equation describing the system is linear having its coefficients as constants.

c) Practically the output i.e. response varies linearly with the input i.e. forcing function for linear systems.

A control system is said to be nonlinear, if,

a. It does not satisfy the principle of superposition.

b. The equations describing the system are nonlinear in nature.

The function $f(x) = x^2$ is nonlinear because

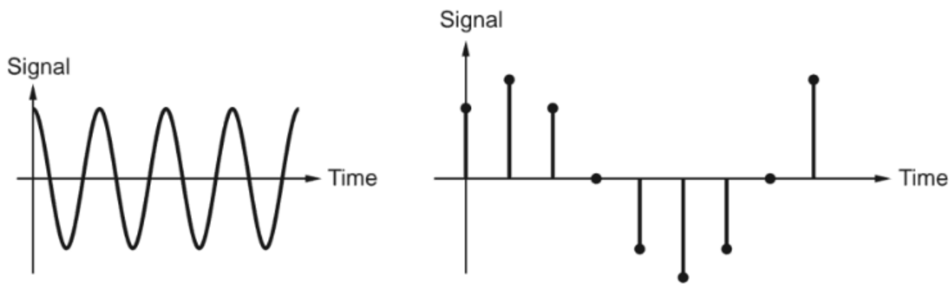
$$f(x_1 + x_2) = (x_1 + x_2)^2 \neq (x_1)^2 + (x_2)^2$$

$$\text{and} \quad f(\alpha\, x) = (\alpha\, x)^2 \neq \alpha\, x^2 \quad \text{where } \alpha = \text{Constant}$$

The equations of nonlinear system involves such nonlinear functions.

c. The output does not vary linearly for nonlinear systems.

6) **Continuous time and discrete time control systems : In a continuous time control system all system variables are the functions of a continuous time variable 't'.** The speed control of a d.c. motor using a tachogenerator feedback is an example of continuous data system. At any time 't' they are dependent on time. **In discrete time systems one or more system variables are known only at certain discrete intervals of time.** They are not continuously dependent on the time.



(a) Continuous signal **(b) Discrete signal**

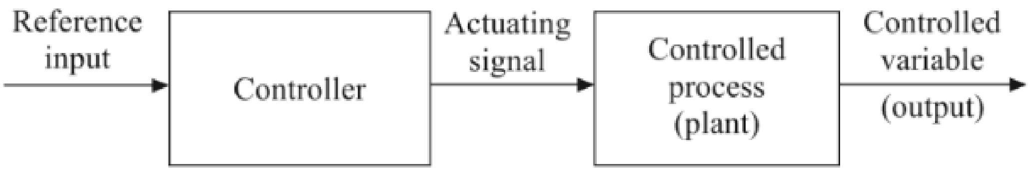
7) **Deterministic and stochastic control systems :** A control system is said to be deterministic when its response to input as well as behaviour to external disturbances is predictable and repeatable. If such response is unpredictable, system is said to be stochastic in nature.

8) **Lumped parameter and distributed parameter control systems :** Control system that can be described by **ordinary differential equations** is called lumped parameter control system. For example, electrical networks with different parameters as resistance, inductance, etc. are lumped parameter systems. Control systems that can be described by **partial differential equations** are called distributed parameter control systems. For example, transmission line having its parameters resistance and inductance totally distributed along it.

9) **Single Input Single Output (SISO) and Multiple Input Multiple Output (MIMO) Systems :** A system having only one input and one output is called single input single output system. For example, a position control system has only one input (desired position) and one output (actual output position). Some systems may have multiple type of inputs and multiple outputs, these are called multiple input multiple output systems.

OPEN-LOOP CONTROL SYSTEM

Those systems in which the output has no effect on the control action, i.e. on the input are called *open-loop control systems*. In other words, in an open-loop control system, the output is neither measured nor fed back for comparison with the input. Open-loop control systems are not feedback systems. Any control system that operates on a time basis is open-loop.



ADVANTAGES:

- These systems are very much suitable for use
- The design of this system is very simple.
- The maintenance aspect of this system is simple.
- The stability is good for some extent of time in this control system
- The convenience in usage is very good
- The cost is low when compared to other systems.
- The output produced is stable

DISADVANTAGES:

- The automatic correction of output deviations cannot be done in this control system
- Inaccuracy exists in these systems
- There exists less bandwidth
- Timely recalibration is required in this control system
- As it is a non-feedback system, the automation of the process will not be initiated
- The disturbances from outside also show the impact on the required output

APPLICATIONS:

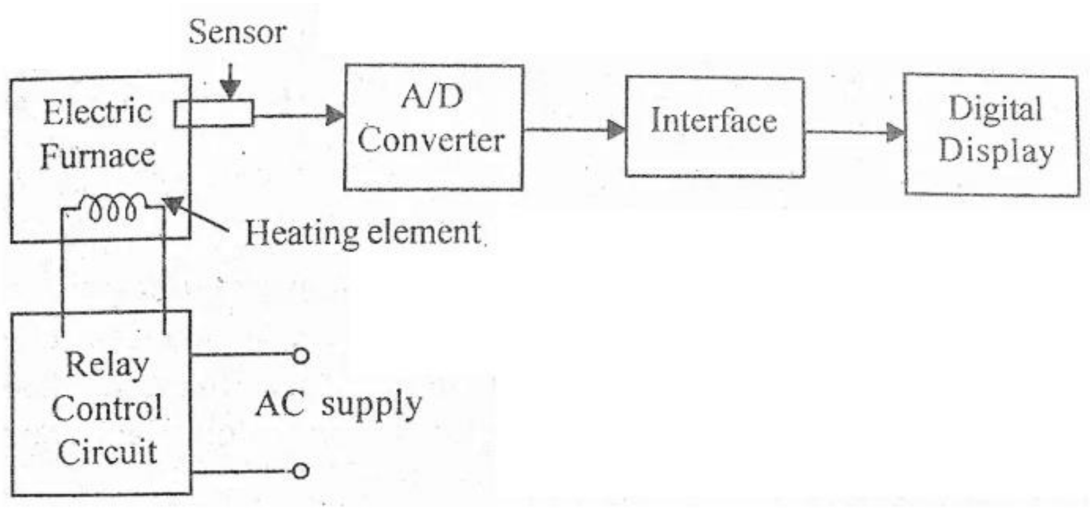
Electric bulb, TV remote control, Washing Machine, Volume on the stereo system, Clothes drier, Servo motor or stepper motor, Door lock systems, Coffee or tea making machine, Inkjet printers etc.

EXAMPLE:

TRAFFIC CONTROL SYSTEM:

Traffic control by means of traffic signals operated on a time basis constitutes an **open-loop control system**. The sequence of control signals is based on a time slot given for each signal. The time slots are decided based on a traffic study. The system will not measure the density of the traffic before giving the signals. Since the time slot does not change according to traffic density, the system is an open-loop system.

TEMPERATURE CONTROL SYSTEM (ELECTRIC FURNACE):



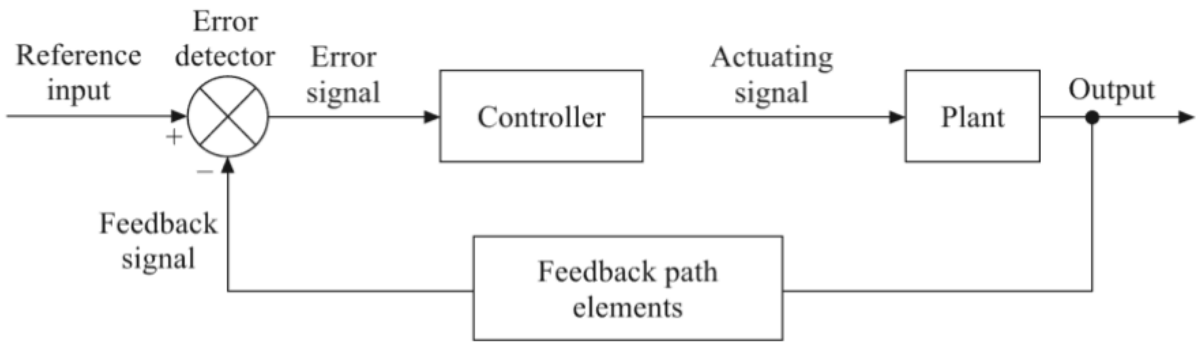
The electric furnace shown below is an open-loop system. The output in the system is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to the heater remains ON. The ON and OFF of the supply are governed by the time setting of the relay.

The temperature is measured by a sensor, which gives art analog voltage corresponding to the temperature of the furnace. The analog signal is converted to a digital signal by an Analog-digital converter (AD converter). The digital signal is given to the digital display device to display the temperature. In this open-loop system, if there is any change in output temperature then the time setting of the relay is not altered automatically.

CLOSED-LOOP CONTROL SYSTEM

Feedback control systems are often referred to as *closed-loop control systems*. In practice, the terms, 'closed-loop control' and 'feedback control' are used interchangeably. In a closed-loop control system, the actuating error signal which is the difference between the input signal and the feedback signal (which may be the output signal itself or a function of the output signal and its derivatives and/or integrals) is fed to the controller so as to reduce the error and bring the output of the system to a desired value. A system that maintains a prescribed relationship between the output and the reference input by comparing them and using the difference as a means of control is called a *feedback control system*. The term 'closed-loop control' always implies the use of feedback control action in order to reduce system error.

The general block diagram of an automatic control system is shown in Figure It consists of an error detector, a controller, a plant and feedback path elements.



The reference input corresponds to desired output. The feedback path elements convert the output to a signal of the same type as that of the reference signal. The feedback signal is proportional to the output signal and is fed to the error detector. The error signal generated by the error detector is the difference between the reference signal and the feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

ADVANTAGES:

- It reduces error in the output by automatically adjusting the system's input.
- This system improves the stability of a system.
- It controls the sensitivity of the system to external factors.
- This control system enhances the robustness of the system.
- It produces reliable and repeatable performance.
- This system also reduces the disturbance compared to the open-loop control system.
- Improved reference tracking

DISADVANTAGES:

- This system many times requires a complex design with more than one feedback path to get the desired outputs.
- The controller becomes unstable and starts to oscillate when the gain of the controller is too sensitive to the changes in its input signal.
- This system is very expensive.

APPLICATIONS:

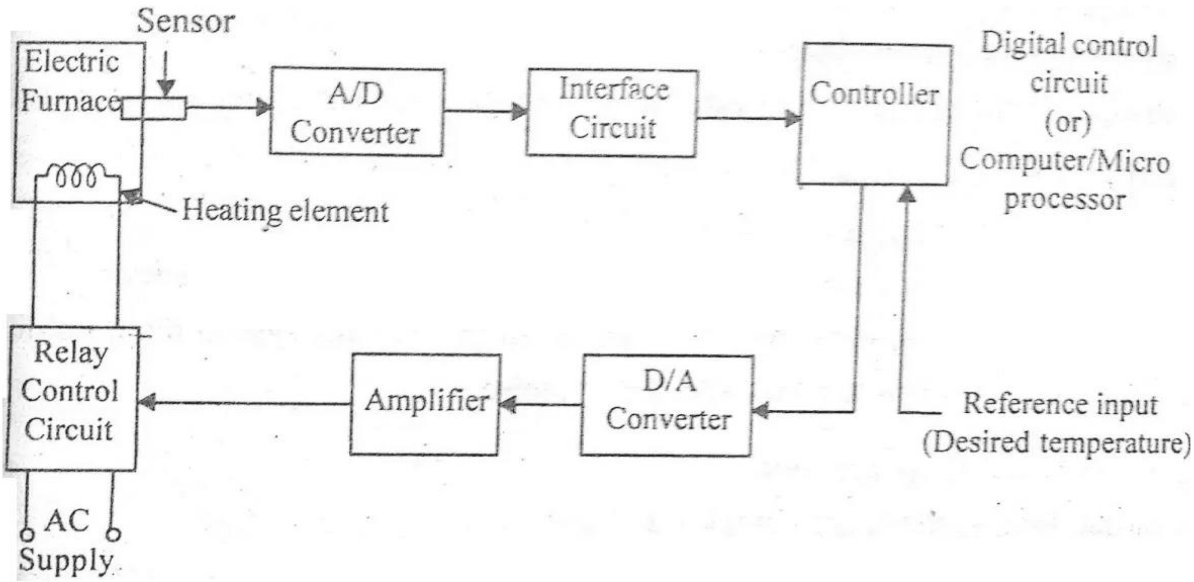
Automatic toaster, Automatic washing machine, Air conditioner, Automatic water level controller in water tanks, DC Motor speed controller, Home heating, Missile launching system, Boiler control, Car's cruise control etc.

EXAMPLE:

TRAFFIC CONTROL SYSTEM:

A traffic control system can be made as a closed-loop system if the time slots of the signals are decided based on the density of traffic. In a closed-loop traffic control system, the density of the traffic is measured on all the sides and the information is fed to a computer. The timings of the control signals are decided by the computer based on the density of traffic Since the closed-loop system dynamically changes the timings, the flow of vehicles will be better than the open-loop system.

TEMPERATURE CONTROL SYSTEM (ELECTRIC FURNACE):



The electric furnace shown in the figure is a closed-loop system example. The output of the closed-loop system is the desired temperature and it depends on the time during which the supply to the heater remains ON.

The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The desired temperature is input to the system through the keyboard or as a signal corresponding to the desired temperature via ports. The actual temperature is sensed by the Sensor and converted to a digital signal by the A/D converter.

The computer reads the actual temperature and compares it with the desired temperature. If it finds any difference then it sends the signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in output. Hence it is a closed-loop system.

COMPARISON BETWEEN OPEN-LOOP AND CLOSED-LOOP CONTROL SYSTEMS

Sl. No.	Open-loop control systems	Closed-loop control systems
1	No feedback is given to the control system	A feedback is given to the control system
2	Cannot be intelligent	Intelligent controlling action
3	There is no possibility of undesirable system oscillation(hunting)	Closed loop control introduces the possibility of undesirable system oscillation(hunting)
4	The output will not very for a constant input, provided the system parameters remain unaltered	In the system the output may vary for a constant input, depending upon the feedback
5	System output variation due to variation in parameters of the system is greater and the output very in an uncontrolled way	System output variation due to variation in parameters of the system is less.
6	Error detection is not present	Error detection is present
7	Small bandwidth	Large bandwidth
8	More stable	Less stable or prone to instability
9	Affected by non-linearities	Not affected by non-linearities
10	Very sensitive in nature	Less sensitive to disturbances
11	Simple design	Complex design
12	Cheap	Costly

MATHEMATICAL MODELING OF CONTROL SYSTEMS

TRANSFER FUNCTION:

Mathematically it is defined as the ratio of Laplace transform of output (response) of the system to the Laplace transform of input (excitation or driving function), under the assumption that ***all initial conditions are zero***.

If T(s) is the transfer function of the system then,

$$T(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}} = \frac{C(s)}{R(s)}$$

The properties of the transfer function are as follows:

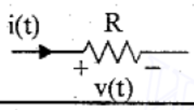
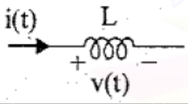
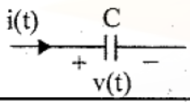
1. The transfer function is defined only for a linear time-invariant system. It is not defined for nonlinear systems.
2. The transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response.

   Alternatively, the transfer function between a pair of input and output variables of a system is the ratio of the Laplace transform of the output to the Laplace transform of the input.
3. All initial conditions of the system are set to zero.
4. The transfer function is independent of the input of the system.
5. The transfer function of a continuous-data system is expressed only as a function of the complex variable s. It is not a function of the real variable time, or any other variable that is used as the independent variable. For discrete-data systems modelled by difference equations, the transfer function is a function of z when the z-transform is used.

MODELING OF ELECTRICAL SYSTEMS

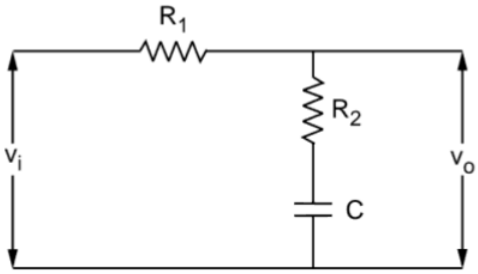
The important elements of an electrical network are R, L and C. The various expressions related to these parameters in time domain and Laplace domain are given in the Table neglecting the initial conditions.

Element	Time domain expression for voltage	Laplace domain expression for voltage	Laplace domain behaviour
Resistance R	$i(t) \times R$	$I(s)R$	R
Industance L	$L \frac{d\,i(t)}{dt}$	$sLI(s)$	sL
Capacitance C	$\frac{1}{C}\int i(t)\,dt$	$\frac{1}{sC} I(s)$	$\frac{1}{sC}$

Element	Voltage across the element	Current through the element
	$v(t) = Ri(t)$	$i(t) = \frac{v(t)}{R}$
	$v(t) = L\frac{d}{dt}i(t)$	$i(t) = \frac{1}{L}\int v(t)\,dt$
	$v(t) = \frac{1}{C}\int i(t)\,dt$	$i(t) = C\frac{dv(t)}{dt}$

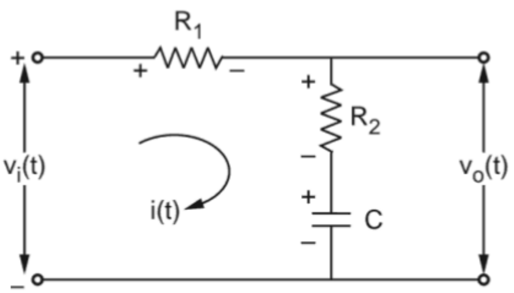
PROBLEMS

1) Determine the transfer function $V_0(s)$ / $V_i(s)$ of the electrical system shown in the fig.



SOL:

The curent flowing is shown in the Fig.



$$v_i(t) = \text{Input}$$

$$v_o(t) = \text{Output}$$

$$\therefore \quad \text{T.F.} = \frac{V_o(s)}{V_i(s)}$$

Applying KVL to the loop,

$$-i(t)\,R_1 - i(t)\,R_2 - \frac{1}{C}\int i(t) + v_i(t) = 0 \qquad \ldots(1)$$

Taking Laplace transform and neglecting initial conditions,

$$I(s)\,R_1 + I(s)\,R_2 + \frac{1}{C}\frac{I(s)}{s} = V_i(s)$$

$$I(s) = \frac{V_i(s)}{R_1 + R_2 + \frac{1}{sC}} = \frac{sC\,V_i(s)}{sC\,(R_1 + R_2) + 1} \quad \ldots(2)$$

The output equation is,

$$v_o(t) = i(t)\,R_2 + \frac{1}{C}\int i(t)\,dt \quad \ldots(3)$$

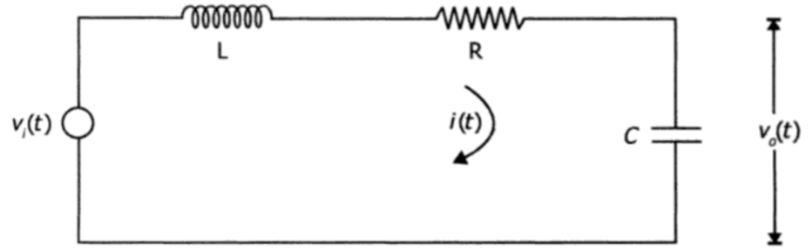
Taking Laplace transform,

$$V_o(s) = I(s)\,R_2 + \frac{1}{C}\frac{I(s)}{s} = I(s)\left[R_2 + \frac{1}{sC}\right] \quad \ldots(4)$$

Using (2) in (4), $V_o(s) = \left\{\frac{sC\,V_i(s)}{sC\,(R_1 + R_2) + 1}\right\}\left[\frac{sC\,R_2 + 1}{sC}\right]$

$$\therefore \quad \text{T.F.} = \frac{V_o(s)}{V_i(s)} = \frac{sC\,R_2 + 1}{sC\,(R_1 + R_2) + 1}$$

2) Determine the transfer function $V_0(s)$ / $V_i(s)$ of the electrical system shown in the fig.



SOL:

The RLC circuit of Fig. is analysed by Kirchhoff's voltage law applied to the closed loop.

The system equation is, $v_i(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt$

Now taking Laplace transform on both sides, we get

$$V_i(s) = LsI(s) + RI(s) + \frac{1}{Cs}I(s)$$

(assuming all initial conditions to be zero)

$$V_i(s) = \left[Ls + R + \frac{1}{Cs}\right]I(s)$$

$$V_i(s) = \left[\frac{LCs^2 + RCs + 1}{Cs}\right]I(s)$$

Let the output voltage $v_o(t)$ be taken across the capacitor, C. Then,

$$v_o(t) = \frac{1}{C}\int idt$$

Taking Laplace transform on both sides of equation we get

$$V_o(s) = \frac{1}{Cs} I(s)$$

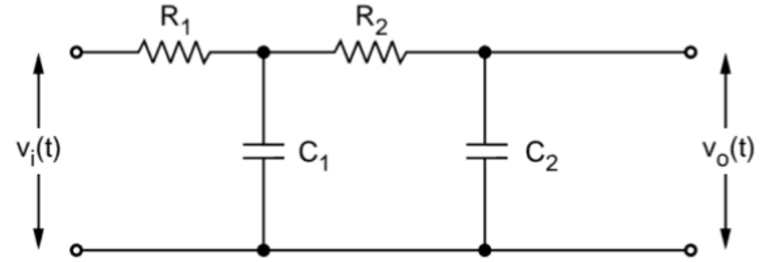
(assuming all initial conditions to be zero)

Therefore, the transfer function is given by

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{(LCs^2 + RCs + 1)} \cdot Cs \frac{I(s)}{I(s)}$$

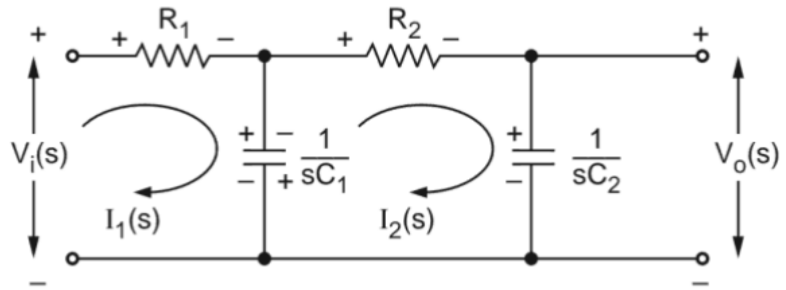
$$\frac{V_o(s)}{V_i(s)} = \frac{1}{(LCs^2 + RCs + 1)}$$

3) Determine the transfer function of an electrical system shown in the fig.



SOL:

The s-domain network of given system is shown in the Fig.



Applying KVL to the two loops,

$$-I_1R_1 - I_1 \times \frac{1}{sC_1} + I_2 \times \frac{1}{sC_1} + V_i(s) = 0 \quad \text{i.e.} \quad I_1\left[R_1 + \frac{1}{sC_1}\right] - I_2 \times \frac{1}{sC_1} = V_i(s) \qquad \text{...(1)}$$

$$-I_2R_2 - I_2 \times \frac{1}{sC_2} - I_2 \times \frac{1}{sC_1} + I_1 \times \frac{1}{sC_1} = 0 \quad \text{i.e.} \quad I_1 \times \frac{1}{sC_1} - I_2 \times \left[R_2 + \frac{1}{sC_1} + \frac{1}{sC_2}\right] = 0 \qquad \text{...(2)}$$

From equation (2),

$$\frac{I_1}{sC_1} = I_2\left[R_2 + \frac{1}{sC_1} + \frac{1}{sC_2}\right] = \frac{I_2\,[sC_1C_2R_2 + C_2 + C_1]}{sC_1C_2}$$

$$\therefore \quad I_1 = \frac{I_2\,[sC_1C_2R_2 + C_2 + C_1]}{C_2} \qquad \text{...(3)}$$

Using equation (3) in (1),

$$\frac{I_2\,[sC_1C_2R_2 + C_1 + C_2]}{C_2} \times \left[R_1 + \frac{1}{sC_1}\right] - I_2 \times \frac{1}{sC_1} = V_i(s)$$

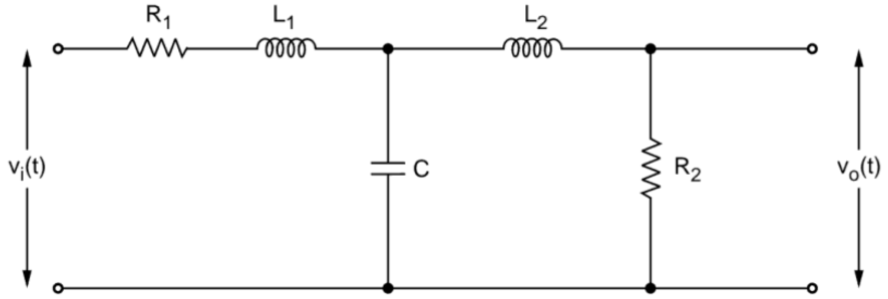
$$\therefore I_2 \left\{ \frac{sC_1C_2R_2 + C_1 + C_2 + s^2C_1^2C_2R_1R_2 + sR_1C_1^2 + sR_1C_1C_2 - C_2}{sC_1C_2} \right\} = V_i(s)$$

But, $V_o(s) = I_2 \times \frac{1}{sC_2}$ i.e. $I_2 = sC_2 V_o(s)$

$$\therefore \quad sC_2 V_o(s)\left\{\frac{sC_1C_2R_2 + C_1 + s^2C_1^2C_2R_1R_2 + sR_1C_1^2 + sR_1C_1C_2}{sC_1C_2}\right\} = V_i(s)$$

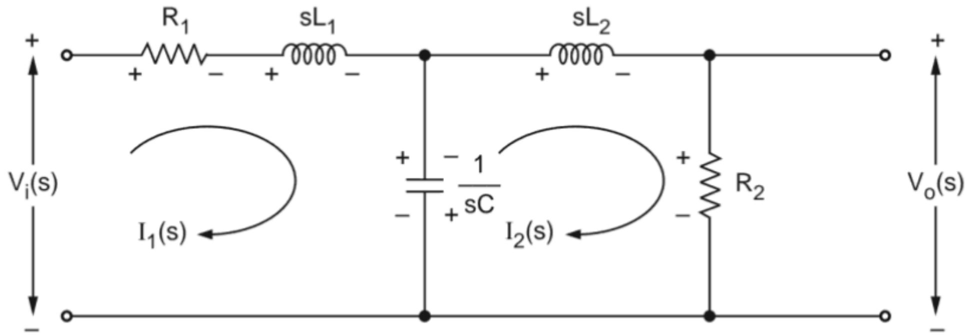
$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{s^2C_1C_2R_1R_2 + s[R_1C_1 + R_2C_2 + R_1C_2] + 1}}$$

4) Determine the transfer function of an electrical system shown in the fig.



SOL:

The s-domain network is shown in the Fig.



Applying KVL to the two loops,

$$-I_1(s)R_1 - sL_1I_1(s) - \frac{1}{sC}I_1(s) + \frac{1}{sC}I_2(s) + V_i(s) = 0$$

i.e.

$$I_1(s)\left[R_1 + sL_1 + \frac{1}{sC}\right] - I_2(s) \times \frac{1}{sC} = V_i(s) \quad \text{...(1)}$$

$$-sL_2I_2(s) - R_2I_2(s) - \frac{1}{sC}I_2(s) + \frac{1}{sC}I_1(s) = 0 \quad \text{i.e.} \quad \frac{1}{sC}I_1(s) = I_2(s)\left[sL_2 + R_2 + \frac{1}{sC}\right]$$

$$\therefore \quad I_1(s) = I_2(s)[s^2L_2C + sR_2C + 1] \quad \text{... (2)}$$

Using in equation (1), $I_2(s)[s^2L_2C + sR_2C + 1]\left[R_1 + sL_1 + \frac{1}{sC}\right] - \frac{1}{sC}I_2(s) = V_i(s)$

$$\therefore \quad I_2(s)\left\{\frac{(s^2L_2C + sR_2C + 1)(sR_1C + s^2L_1C + 1)}{sC} - \frac{1}{sC}\right\} = V_i(s)$$

$$\therefore \quad I_2(s)\left\{\frac{(s^2L_2C + sR_2C + 1)(sR_1C + s^2L_1C + 1) - 1}{sC}\right\} = V_i(s) \quad \text{...(3)}$$

But $V_o(s) = I_2(s)R_2$ i.e. $I_2(s) = \frac{V_o(s)}{R_2}$

$$\frac{V_o(s)}{R_2}\left\{\frac{(s^2L_2C+sR_2C+1)(sR_1C+s^2L_1C+1)-1}{sC}\right\} = V_i(s)$$

$$\frac{V_o(s)}{V_i(s)} = \left\{\frac{sR_2C}{(s^2L_2C+sR_2C+1)(sR_1C+s^2L_1C+1)-1}\right\}$$

$$= \frac{R_2C}{(s^3L_1L_2C^2+s^2(L_1R_2C^2+L_2R_1C^2)+s(L_1C+R_1R_2C^2+L_2C)+(R_1C+R_2C)}$$

5) The dynamic behaviour of the system is described by the differential equation

$$\frac{dc}{dt} + 10c = 10e$$

Where 'e' the input and 'c' is the output. Determine the transfer function of the system.

SOL:

Given differential eq. $\frac{dc}{dt} + 10c = 10e$

Taking laplace Transform on both sides

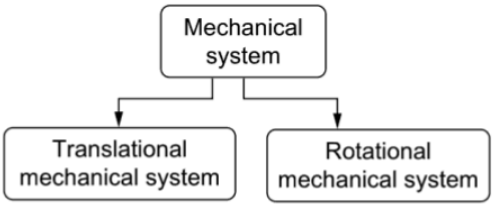
$$S\,C(S) + 10\,C(S) = 10\,E(S)$$

$$C(S)\,(S+10) = 10\,E(S)$$

$\therefore$ Transfer function $\frac{C(S)}{E(S)} = \frac{10}{S+10}$

MODELING OF MECHANICAL SYSTEMS

The general classification of mechanical system is of two types: (i) translational and (ii) rotational as shown in Fig.



TRANSLATIONAL MECHANICAL SYSTEM

Consider a mechanical system in which motion is taking place along a straight line. Such systems are of translational type. These systems are characterised by displacement, linear velocity and linear acceleration.

According to Newton's law of motion, sum of forces applied on rigid body or system must be equal to sum of forces consumed to produce displacement, velocity and acceleration in various elements of the system.

The following elements are dominantly involved in the analysis of translational motion systems.

i) Mass ii) Spring iii) Friction.

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m

$v = \frac{dx}{dt}$ = Velocity, m/sec

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

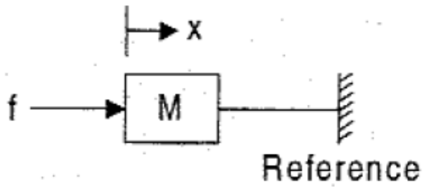
Consider an ideal mass element shown in fig which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.

Let, f = Applied force

f_m = Opposing force due to mass

Here, $f_m \propto \dfrac{d^2x}{dt^2}$ or $f_m = M\dfrac{d^2x}{dt^2}$

By Newton's second law, $\boxed{f = f_m = M\dfrac{d^2x}{dt^2}}$



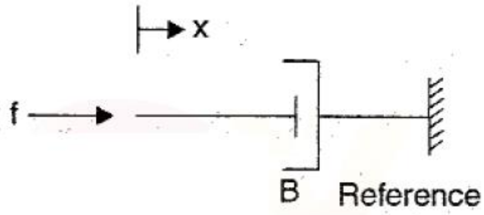
Consider an ideal frictional element dashpot shown in fig which has negligible mass and elasticity . Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.

Let, f = Applied force

f_b = Opposing force due to friction

Here, $f_b \propto \dfrac{dx}{dt}$ or $f_b = B\dfrac{dx}{dt}$

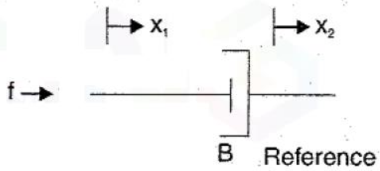
By Newton's second law, $\boxed{f = f_b = B\dfrac{dx}{dt}}$



When the dashpot has displacement at both ends as shown in fig , the opposing force is proportional to differential velocity.

$f_b \propto \dfrac{d}{dt}(x_1 - x_2)$ or $f_b = B\dfrac{d}{dt}(x_1 - x_2)$

$\therefore \boxed{f = f_b = B\dfrac{d}{dt}(x_1 - x_2)}$



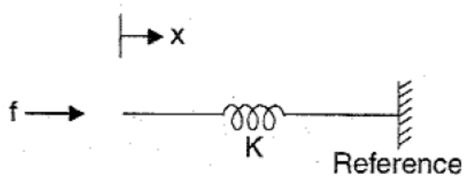
Consider an ideal elastic element spring shown in fig which has negligible mass and friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

Let, f = Applied force

f_k = Opposing force due to elasticity

Here $f_k \propto x$ or $f_k = K\,x$

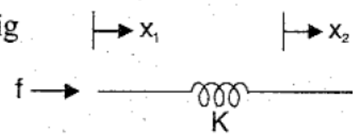
By Newton's second law, $\boxed{f = f_k = Kx}$



When the spring has displacement at both ends as shown in fig the opposing force is proportional to differential displacement.

$f_k \propto (x_1 - x_2)$ or $f_k = K(x_1 - x_2)$

$\therefore \boxed{f = f_k = K(x_1 - x_2)}$



ROTATIONAL MECHANICAL SYSTEM

The model of rotational mechanical systems can be obtained by using three elements, ***moment of inertia*** [J] of mass, ***dash-pot*** with rotational frictional coefficient [B] and ***torsional spring*** with stiffness [K].

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by ***Newton's second law of motion*** for rotational systems. It states that the sum of torques acting on a body is zero (or Newton's law states that the sum of applied torques is equal to the sum of opposing torques on a body).

LIST OF SYMBOLS USED IN MECHANICAL ROTATIONAL SYSTEM

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec2

T = Applied torque, N-m

J = Moment of inertia, Kg-m2/rad

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad

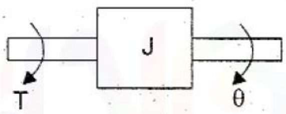
TORQUE BALANCE EQUATIONS OF IDEALISED ELEMENTS

Consider an ideal mass element shown in fig which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

Let, T = Applied torque.

T_j = Opposing torque due to moment of inertia of the body.

Here $T_j \propto \frac{d^2\theta}{dt^2}$ or $T_j = J\frac{d^2\theta}{dt^2}$



By Newton's second law,

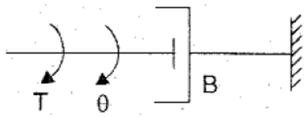
$$\boxed{T = T_j = J\frac{d^2\theta}{dt^2}}$$

Consider an ideal frictional element dash pot shown in fig which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque which is proportional to the angular velocity of the body.

Let, T = Applied torque.

T_b = Opposing torque due to friction.

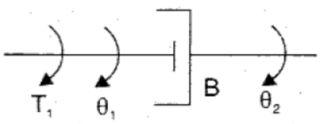
$T_b \propto \frac{d\theta}{dt}$ or $T_b = B\frac{d\theta}{dt}$



By Newton's second law, $\boxed{T = T_b = B\frac{d\theta}{dt}}$

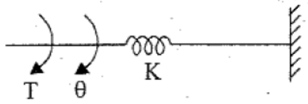
When the dash pot has angular displacement at both ends as shown in fig the opposing torque is proportional to the differential angular velocity.

$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2)$ or $T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$



$$\boxed{\therefore T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)}$$

Consider an ideal elastic element, torsional spring as shown in fig which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.



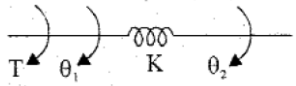
Let, T = Applied torque.

T_k = Opposing torque due to elasticity.

$T_k \propto \theta$ or $T_k = K\theta$

By Newton's second law, $\boxed{T = T_k = K\theta}$

When the spring has angular displacement at both ends as shown in fig the opposing torque is proportional to differential angular displacement.

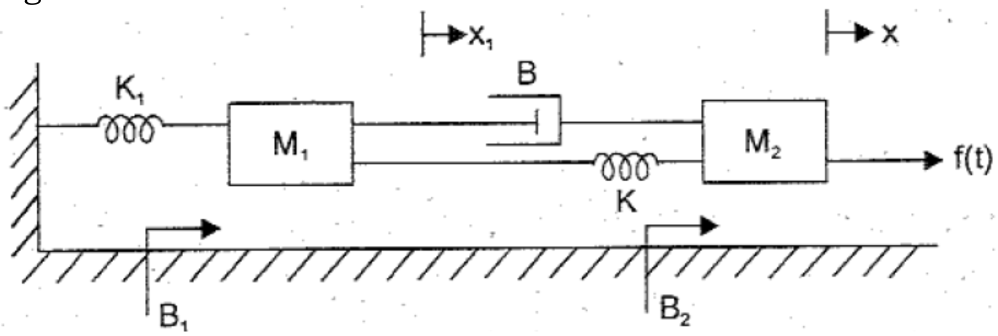


$T_k \propto (\theta_1 - \theta_2)$ or $T_k = K(\theta_1 - \theta_2)$

$\therefore \boxed{T = T_k = K(\theta_1 - \theta_2)}$

PROBLEMS

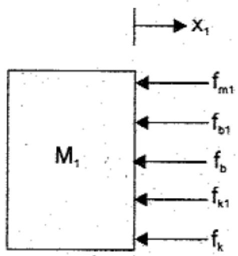
1) Write the differential equations governing the mechanical system shown in the fig. and determine the transfer function



SOL:

In the given system, applied force 'f(t)' is the input and displacement 'x' is the output.

Let the displacement of mass M_1 be x_1. The free body diagram of mass M_1 is shown in fig . The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{k1} and f_k.



$$f_{m1} = M_1 \frac{d^2x_1}{dt^2}; \quad f_{b1} = B_1 \frac{dx_1}{dt}; \quad f_{k1} = K_1 x_1;$$

$$f_b = B\frac{d}{dt}(x_1 - x); \quad f_k = K(x_1 - x)$$

By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

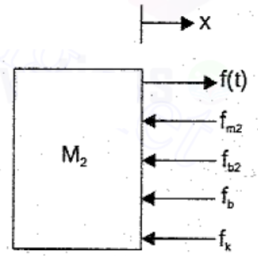
$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs\,[X_1(s) - X(s)] + K_1 X_1(s) + K\,[X_1(s) - X(s)] = 0$$

$$X_1(s)\,[M_1 s^2 + (B_1 + B)s + (K_1 + K)] - X(s)\,[Bs + K] = 0$$

$$X_1(s)\,[M_1 s^2 + (B_1 + B)s + (K_1 + K)] = X(s)\,[Bs + K]$$

$$\therefore X_1(s) = X(s) \frac{Bs + K}{M_1 s^2 + (B_1 + B)\,s + (K_1 + K)} \quad \text{......(1)}$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are marked as f_{m2}, f_{b2}, f_b and f_k.



$$f_{m2} = M_2 \frac{d^2x}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}(x - x_1) \quad ; \quad f_k = K(x - x_1)$$

By Newton's second law,

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_2s^2X(s) + B_2sX(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)\,[M_2s^2 + (B_2 + B)s + K] - X_1(s)[Bs + K] = F(s) \quad ....(2)$$

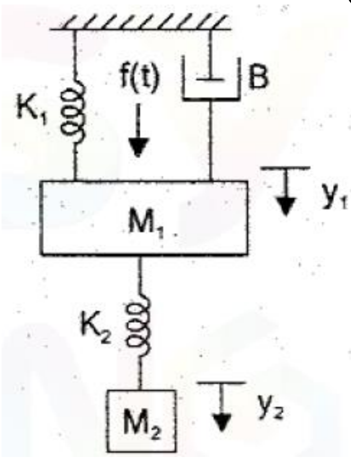
Substituting for $X_1(s)$ from equation (1) in equation (2) we get,

$$X(s)\,[M_2s^2 + (B_2 + B)s + K] - X(s)\frac{(Bs + K)^2}{M_1s^2 + (B_1 + B)s + (K_1 + K)} = F(s)$$

$$X(s)\left[\frac{[M_2s^2 + (B_2 + B)s + K]\,[M_1s^2 + (B_1 + B)s + (K_1 + K)] - (Bs + K)^2}{M_1s^2 + (B_1 + B)\,s + (K_1 + K)}\right] = F(s)$$

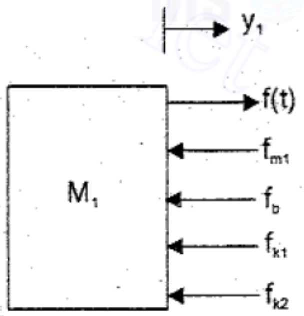
$$\therefore \frac{X(s)}{F(s)} = \frac{M_1s^2 + (B_1 + B)s + (K_1 + K)}{[M_1s^2 + (B_1 + B)s + (K_1 + K)]\,[M_2s^2 + (B_2 + B)s + K] - (Bs + K)^2}$$

2) Write the differential equations governing the mechanical system shown in the fig. and determine the transfer function $Y_2(s)$ / $F(s)$.



SOL:

The free body diagram of mass M_1 is shown in fig



The opposing forces are marked as f_{m1}, f_b, f_{k1} and f_{k2}

$$f_{m1} = M_1 \frac{d^2y_1}{dt^2} \;;\; f_b = B\frac{dy_1}{dt} \;;\; f_{k1} = K_1y_1 \;;\; f_{k2} = K_2(y_1 - y_2)$$

By Newton's second law, $f_{m1} + f_b + f_{k1} + f_{k2} = f(t)$

$$\therefore M_1\frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_1y_1 + K_2(y_1 - y_2) = f(t) \qquad .....(1)$$

On taking Laplace transform of equation (1) with zero initial condition we get,

$$M_1s^2Y_1(s) + BsY_1(s) + K_1Y_1(s) + K_2[Y_1(s) - Y_2(s)] = F(s)$$

$$Y_1(s)[M_1s^2 + Bs + (K_1 + K_2)] - Y_2(s)K_2 = F(s) \qquad .....(2)$$

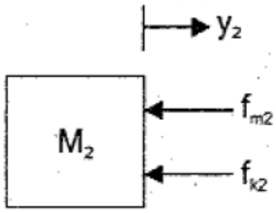
The free body diagram of mass M_2 is shown in fig

The opposing forces acting on M_2 are f_{m2} and f_{k2}.

$$f_{m2} = M_2\frac{d^2y_2}{dt^2} \;; \qquad f_{k2} = K_2(y_2 - y_1)$$

By Newton's second law, $f_{m2} + f_{k2} = 0$

$$\therefore M_2\frac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0$$



On taking Laplace transform of above equation we get,

$$M_2s^2Y_2(s) + K_2[Y_2(s) - Y_1(s)] = 0$$

$$Y_2(s)\,[M_2s^2 + K_2] - Y_1(s)\,K_2 = 0$$

$$\therefore Y_1(s) = Y_2(s)\,\frac{M_2s^2 + K_2}{K_2} \qquad .....(3)$$

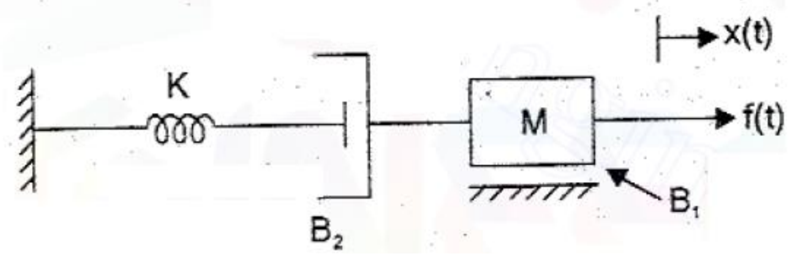
Substituting for $Y_1(s)$ from equation (3) in equation (2) we get,

$$Y_2(s)\left[\frac{M_2s^2 + K_2}{K_2}\right]\left[M_1s^2 + Bs + (K_1 + K_2)\right] - Y_2(s)\,K_2 = F(s)$$

$$Y_2(s)\left[\frac{(M_2s^2 + K_2)\,[M_1s^2 + Bs + (K_1 + K_2)] - K_2^2}{K_2}\right] = F(s)$$

$$\therefore \frac{Y_2(s)}{F(s)} = \frac{K_2}{\left[M_1s^2 + Bs + (K_1 + K_2)\right]\left[M_2s^2 + K_2\right] - K_2^2}$$

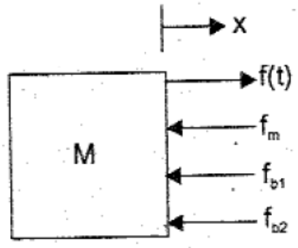
3) Write the differential equations governing the mechanical system shown in the fig. and determine the transfer function X(s) / F(s).



SOL:

The free body diagram of mass M is shown in fig . The opposing forces are marked as f_m, f_{b1} and f_{b2}.

$$f_m = M\frac{d^2x}{dt^2} \; ; \; f_{b1} = B_1\frac{dx}{dt} \; ; \; f_{b2} = B_2\frac{d}{dt}(x - x_1)$$



By Newton's second law the force balance equation is,

$$f_m + f_{b1} + f_{b2} = f(t)$$

$$\therefore M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B_2\frac{d}{dt}(x - x_1) = f(t)$$

On taking Laplace transform of the above equation we get,

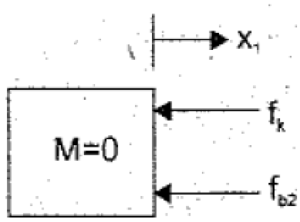
$$Ms^2 X(s) + B_1 s X(s) + B_2 s [X(s) - X_1(s)] = F(s)$$

$$\left[Ms^2 + (B_1 + B_2) s\right] X(s) - B_2 s \; X_1(s) = F(s) \quad .....(1)$$

The free body diagram at the meeting point of spring and dashpot is shown in fig The opposing forces are marked as f_k and f_{b2}.

$$f_{b2} = B_2\frac{d}{dt}(x_1 - x) \; ; \quad f_k = K x_1$$

By Newton's second law, $f_{b2} + f_k = 0$



$$\therefore B_2\frac{d}{dt}(x_1 - x) + K x_1 = 0$$

On taking Laplace transform of the above equation we get,

$$B_2 s [X_1(s) - X(s)] + K X_1(s) = 0$$

$$(B_2 s + K) X_1(s) - B_2 s X(s) = 0$$

$$\therefore X_1(s) = \frac{B_2 s}{B_2 s + K} X(s) \quad .....(2)$$

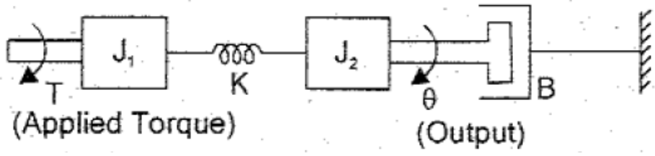
Substituting for $X_1(s)$ from equation (2) in equation (1) we get,

$$\left[M s^2 + (B_1 + B_2) s\right] X(s) - B_2 s\left[\frac{B_2 s}{B_2 s + K}\right] X(s) = F(s)$$

$$X(s) \frac{\left[[M s^2 + (B_1 + B_2) s] (B_2 s + K) - (B_2 s)^2\right]}{B_2 s + K} = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{B_2 s + K}{[M s^2 + (B_1 + B_2) s] (B_2 s + K) - (B_2 s)^2}$$

4) Write the differential equations governing the mechanical system shown in the fig. and determine the transfer function.

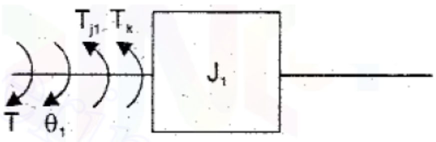


SOL:

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig

The opposing torques acting on J_1 are marked as T_{j1} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_k = K(\theta_1 - \theta)$$



By Newton's second law, $T_{j1} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T \qquad .....(1)$$

On taking Laplace transform of equation (1) with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

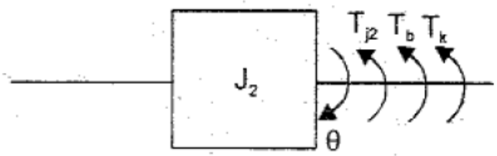
$$(J_1 s^2 + K)\,\theta_1(s) - K\,\theta(s) = T(s) \qquad .....(2)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig

The opposing torques acting on J_2 are marked as T_{j2}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

By Newton's second law, $T_{j2} + T_b + T_k = 0$



$$\therefore J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2 s^2\theta(s) + B\,s\,\theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$(J_2 s^2 + Bs + K)\,\theta(s) - K\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) \qquad .....(3)$$

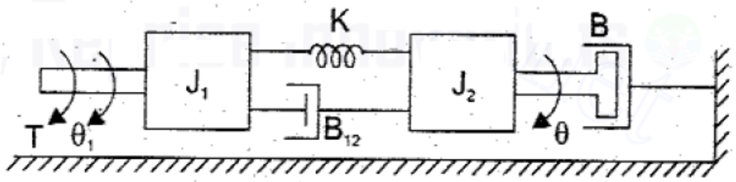
Substituting for $\theta_1(s)$ from equation (3) in equation (2) we get,

$$(J_1 s^2 + K)\,\frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) - K\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$$

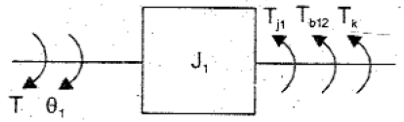
$$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

5) Write the differential equations governing the mechanical system shown in the fig. and determine the transfer function $\boldsymbol{\theta}$(s) / T(s).



SOL:

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig The opposing torques acting on J_1 are marked as T_{j1}, T_{b12} and T_k.



$$T_{j1} = J_1\frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta_1 - \theta) \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_{b12} + T_k = T$

$$J_1\frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$$

On taking Laplace transform of above equation with zero initial conditions we get,

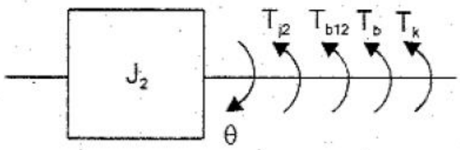
$$J_1s^2\theta_1(s) + s\,B_{12}\,[\theta_1(s) - \theta(s)] + K\theta_1(s) - K\theta(s) = T(s)$$

$$\theta_1(s)\,[J_1s^2 + sB_{12} + K] - \theta(s)\,[sB_{12} + K] = T(s) \qquad .....(1)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig
The opposing torques are marked as T_{j2}, T_{b12}, T_b and T_k.

$$T_{j2} = J_2\frac{d^2\theta}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta - \theta_1)$$

$$T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$



By Newton's second law, $T_{j2} + T_{b12} + T_b + T_k = 0$

$$J_2\frac{d^2\theta}{dt^2} + B_{12}\frac{d}{dt}(\theta - \theta_1) + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2\frac{d^2\theta}{dt^2} - B_{12}\frac{d\theta_1}{dt} + \frac{d\theta}{dt}(B_{12} + B) + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2s^2\theta(s) - B_{12}s\theta_1(s) + s\theta(s)\,[B_{12} + B] + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)\,[s^2J_2 + s(B_{12} + B) + K] - \theta_1(s)\,[sB_{12} + K] = 0$$

$$\theta_1(s) = \frac{[s^2J_2 + s(B_{12} + B) + K]}{[sB_{12} + K]}\,\theta(s) \qquad ....(2)$$

Substituting for $\theta_1(s)$ from equation (2) in equation (1) we get,

$$[J_1s^2 + sB_{12} + K]\,\frac{[J_2s^2 + s(B_{12} + B) + K]\,\theta(s)}{(sB_{12} + K)} - (sB_{12} + K)\,\theta(s) = T(s)$$

$$\left[\frac{(J_1s^2 + sB_{12} + K)\,[J_2s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}{(sB_{12} + K)}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_1s^2 + sB_{12} + K)\,[J_2s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}$$

ANALOGOUS SYSTEMS

In between electrical and mechanical systems there exists a fixed analogy and their exists a similarity between their equilibrium equations. Due to this, it is possible to draw an electrical system which will behave exactly similar to the given mechanical system, this is called electrical analogous of given mechanical system and vice versa. It is always advantageous to obtain **electrical analogous** of the given mechanical system as we are well familiar with the methods of analysing electrical network than mechanical systems.

There are two methods of obtaining electrical analogous networks, namely

1) Force - Voltage Analogy i.e. Direct Analogy.

2) Force - Current Analogy i.e. Inverse Analogy.

FORCE – VOLTAGE (OR) TORQUE – VOLTAGE ANALOGY (MESH ANALYSIS):

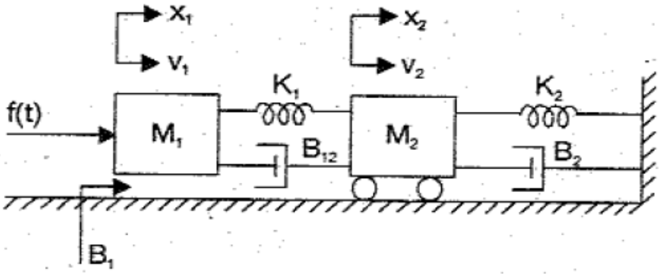
Translational	Rotational	Electrical
Force	Torque T	Voltage V
Mass M	Inertia J	Inductance L
Friction constant B	Tortional friction constant B	Resistance R
Spring constant K N/m	Tortional spring constant K Nm/rad	Reciprocal of capacitor 1/C
Displacement 'x'	θ	Charge q
Velocity $\dot{x} = \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Current $i = \frac{dq}{dt}$

FORCE – CURRENT (OR) TORQUE – CURRENT ANALOGY (NODAL ANALYSIS):

Translational	Rotational	Electrical
F Force	T	Current I
M Mass	J	C
B friction	B	1/R
K Spring	K	1/L
x displacement	θ	ϕ
$\dot{x}$ Velocity $= \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Voltage 'e' $= \frac{d\phi}{dt}$

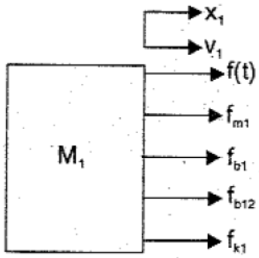
PROBLEMS

1) Write the differential equations governing the mechanical system shown in the fig. Draw the force-voltage and force-current electrical analogous circuits and verify by writing the mesh and nodal equations.



SOL:

The free body diagram of M_1 is shown in fig The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.



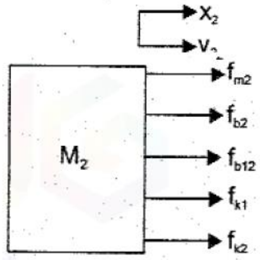
$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2) \quad ; \quad f_{k1} = K_1(x_1 - x_2)$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \quad \text{......(1)}$$

The free body diagram of M_2 is shown in fig .The opposing forces are marked as f_{m2}, f_{b2}, f_{b12}, f_{k1} and f_{k2}



$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx_2}{dt} \quad ; \quad f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k1} = K_1(x_2 - x_1) \quad ; \quad f_{k2} = K_2 x_2$$

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b12} + f_{k1} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = 0 \quad \text{.....(2)}$$

On replacing the displacements by velocity in the differential equations (1) and (2) of the mechanical system we get,

$$\left(\text{i.e, } \frac{d^2x}{dt^2} = \frac{dv}{dt} \quad ; \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt \right)$$

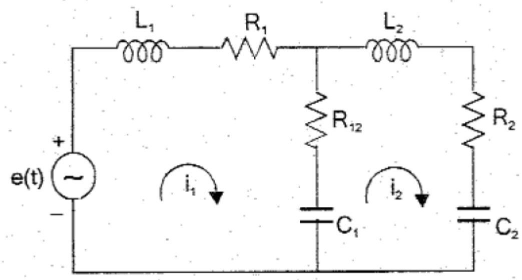
$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1 \int (v_1 - v_2)\,dt = f(t) \quad \text{.....(3)}$$

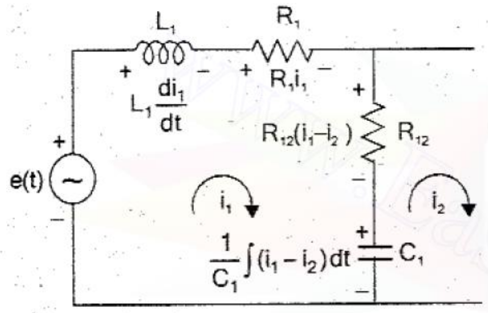
$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_2 \int v_2\,dt + B_{12}(v_2 - v_1) + K_1 \int (v_2 - v_1)\,dt = 0 \quad \text{.....(4)}$$

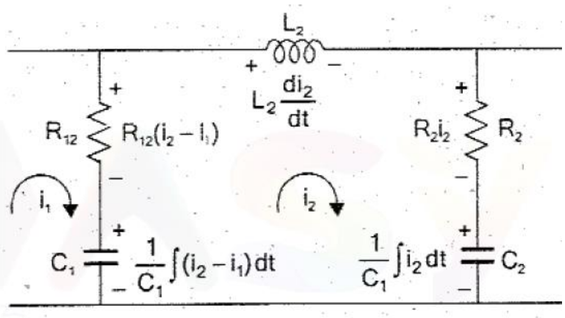
F-V ANALOGOUS CIRCUIT:

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \to e(t)$	$M_1 \to L_1$	$B_1 \to R_1$	$K_1 \to 1/C_1$
$v_1 \to i_1$	$M_2 \to L_2$	$B_2 \to R_2$	$K_2 \to 1/C_2$
$v_2 \to i_2$		$B_{12} \to R_{12}$	







The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig are given below

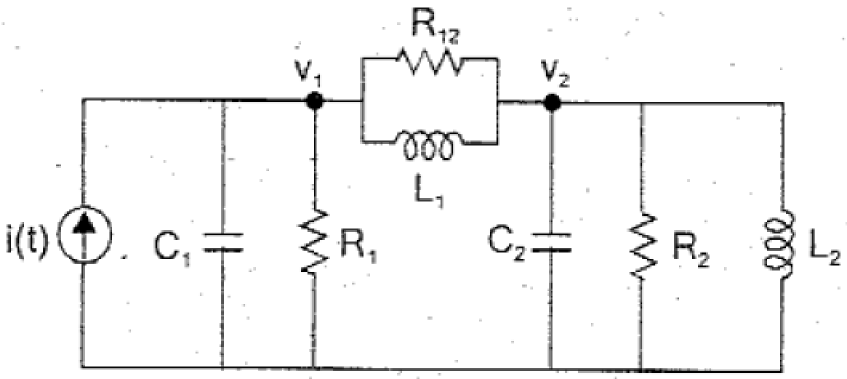
$$L_1\frac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int(i_1 - i_2)dt = e(t) \qquad .....(5)$$

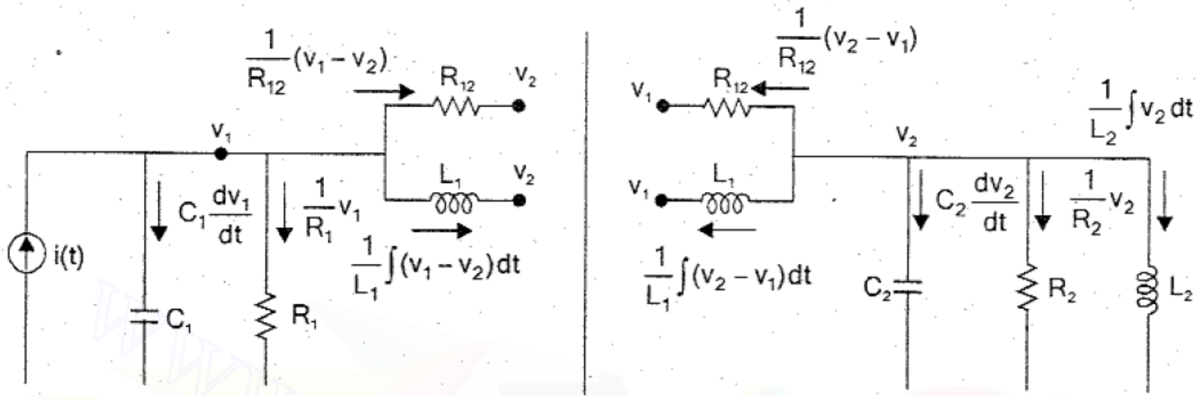
$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\, dt + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int(i_2 - i_1)dt = 0 \qquad .....(6)$$

F-I ANALOGOUS CIRCUIT:

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$v_1 \rightarrow v_1$	$M_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_2 \rightarrow 1/L_2$
$v_2 \rightarrow v_2$	$B_{12} \rightarrow 1/R_{12}$		



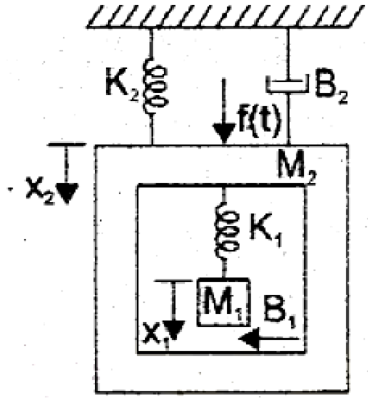


The node basis equations using Kirchoff's current law for the circuit shown in fig are given below

$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}v_1 + \frac{1}{R_{12}}(v_1 - v_2) + \frac{1}{L_1}\int(v_1 - v_2)dt = i(t) \qquad .....(7)$$

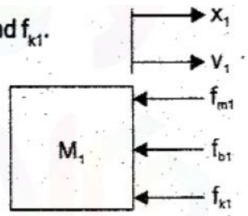
$$C_2\frac{dv_2}{dt} + \frac{1}{R_2}v_2 + \frac{1}{L_2}\int v_2 dt + \frac{1}{R_{12}}(v_2 - v_1) + \frac{1}{L_1}\int(v_2 - v_1)dt = 0 \qquad .....(8)$$

2) Write the differential equations governing the mechanical system shown in the fig. Draw the force-voltage and force-current electrical analogous circuits and verify by writing the mesh and nodal equations.



SOL:

The free body diagram of M_1 is shown in fig The opposing forces are marked as f_{m1}, f_{b1} and f_{k1}.

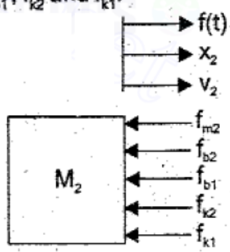


$$f_{m1} = M_1\frac{d^2x_1}{dt^2}\ ;\ f_{b1} = B_1\frac{d(x_1 - x_2)}{dt}\ ;\ f_{k1} = K_1(x_1 - x_2)$$

By Newton's second law, $f_{m1} + f_{b1} + f_{k1} = 0$

$$M_1\frac{d^2x_1}{dt^2} + B_1\frac{d(x_1 - x_2)}{dt} + K_1(x_1 - x_2) = 0 \qquad .....(1)$$

The free body diagram of M_2 is shown in fig The opposing forces are marked as f_{m2}, f_{b2}, f_{b1}, f_{k2} and f_{k1}.



$$f_{m2} = M_2\frac{d^2x_2}{dt^2}\ ;\ f_{b2} = B_2\frac{dx_2}{dt}\ ;\ f_{b1} = B_1\frac{d}{dt}(x_2 - x_1)$$

$$f_{k2} = K_2x_2\ ;\ f_{k1} = K_1(x_2 - x_1)$$

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b1} + f_{k1} = f(t)$

$$M_2\frac{d^2x_2}{dt^2} + B_2\frac{dx_2}{dt} + K_2x_2 + B_1\frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = f(t) \qquad .....(2)$$

On replacing the displacements by velocity in the differential equations (1) and (2) governing the mechanical system

$$\left(\text{i.e., } \frac{d^2x}{dt^2} = \frac{dv}{dt},\ \ \frac{dx}{dt} = v \text{ and } x = \int v\,dt\right)$$

$$M_1\frac{dv_1}{dt} + B_1(v_1 - v_2) + K_1\int(v_1 - v_2)dt = 0 \qquad .....(3)$$

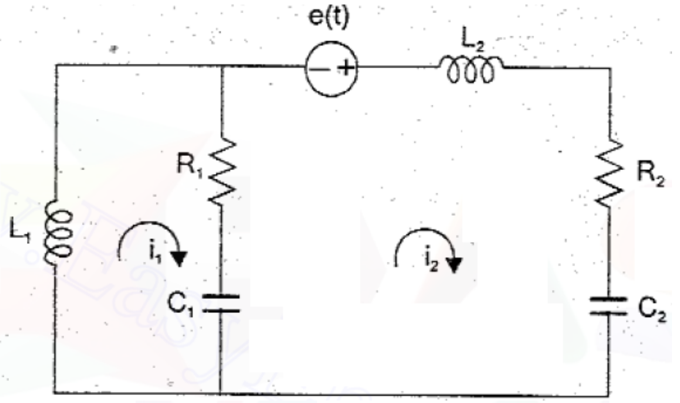
$$M_2\frac{dv_2}{dt} + B_2v_2 + K_2\int v_2dt + B_1(v_2 - v_1) + K_1\int(v_2 - v_1)dt = f(t) \qquad .....(4)$$

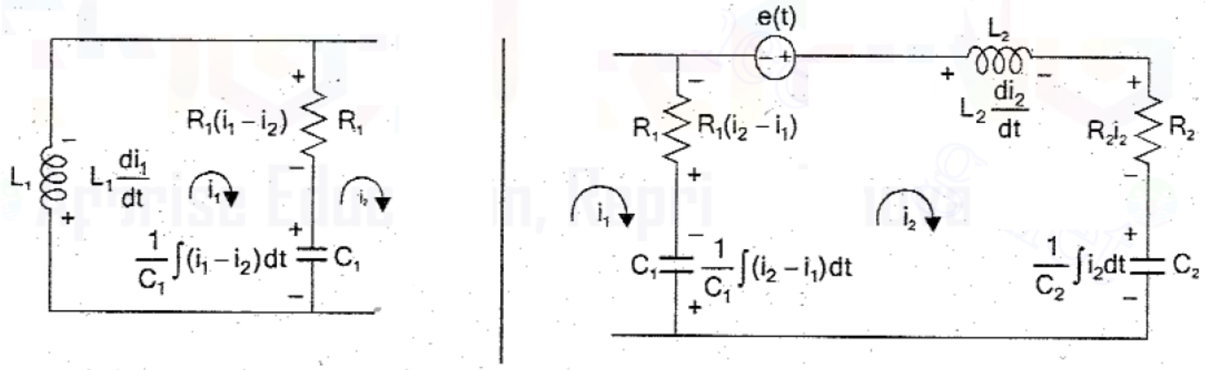
F-V ANALOGOUS CIRCUIT:

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow e(t)$	$v_1 \rightarrow i_1$	$M_1 \rightarrow L_1$	$K_1 \rightarrow 1/C_1$	$B_1 \rightarrow R_1$
	$v_2 \rightarrow i_2$	$M_2 \rightarrow L_2$	$K_2 \rightarrow 1/C_2$	$B_2 \rightarrow R_2$

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig . are given below,





$$L_1\frac{di_1}{dt}+R_1(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)\,dt=0$$

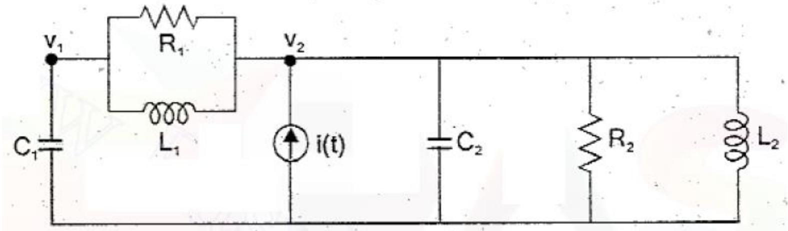
$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2\,dt+\frac{1}{C_1}\int(i_2-i_1)\,dt+R_1(i_2-i_1)=e(t)$$

F-I ANALOGOUS CIRCUIT:

The electrical analogous elements for the elements of mechanical system are given below.

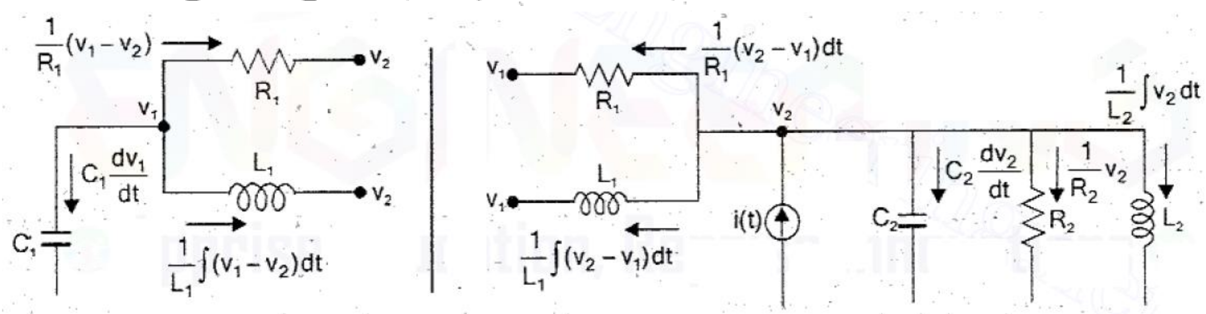
$f(t)\rightarrow i(t)$	$v_1\rightarrow v_1$	$M_1\rightarrow C_1$	$B_1\rightarrow 1/R_1$	$K_1\rightarrow 1/L_1$
	$v_2\rightarrow v_2$	$M_2\rightarrow C_2$	$B_2\rightarrow 1/R_2$	$K_2\rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig. , are given below,

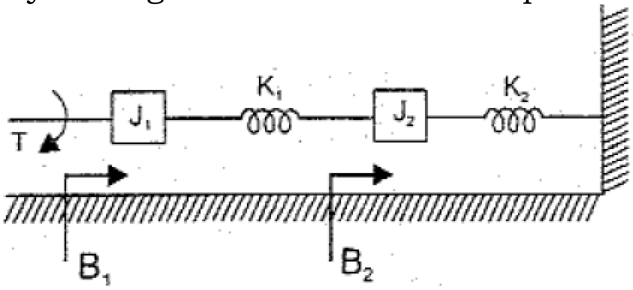


$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)\,dt=0$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_1}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=i(t)$$

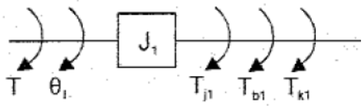


3) Write the differential equations governing the mechanical system shown in the fig. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing the mesh and nodal equations.



SOL:

The free body diagram of J_1 is shown in fig The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

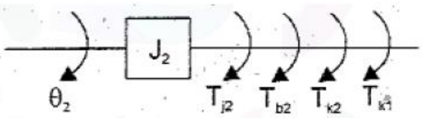


$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \;;\; T_{b1} = B_1 \frac{d\theta_1}{dt} \;;\; T_{k1} = K_1(\theta_1 - \theta_2)$$

By Newton's second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) = T \qquad .....(1)$$

The free body diagram of J_2 is shown in fig . The opposing torques are marked as T_{j2}, T_{b2}, T_{k2} and T_{k1}.



$$T_{j2} = J_2 \frac{d^2\theta_2}{dt^2} \;;\; T_{b2} = B_2 \frac{d\theta_2}{dt}$$

$$T_{k2} = K_2\theta_2 \;;\; T_{k1} = K_1(\theta_2 - \theta_1)$$

By Newton's second law, $T_{j2} + T_{b2} + T_{k2} + T_{k1} = 0$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2) + K_1(\theta_2 - \theta_1) = 0 \qquad .....(2)$$

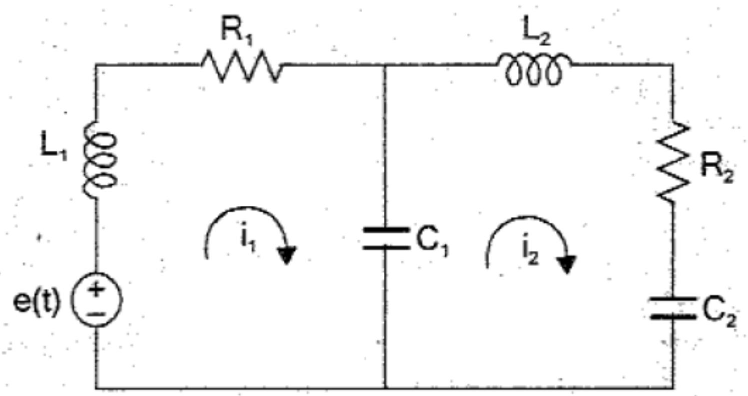
On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

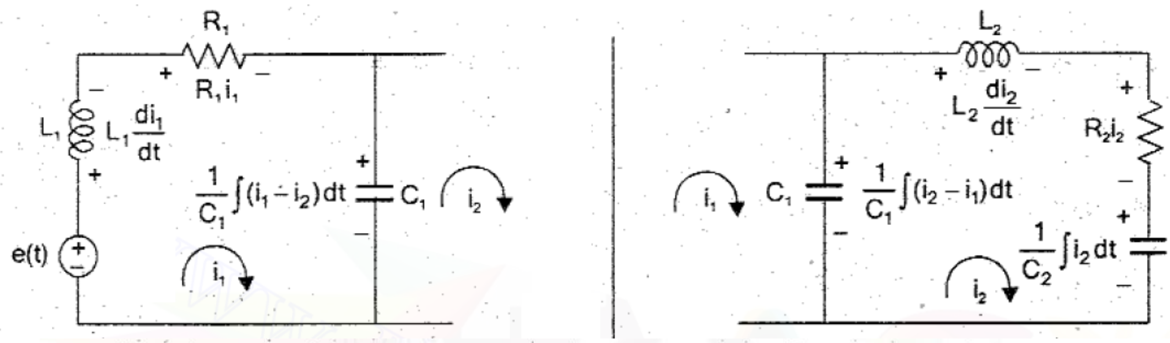
$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \;;\; \frac{d\theta}{dt} = \omega \text{ and } \theta = \int \omega \, dt\right)$$

$$J_1 \frac{d\omega_1}{dt} + B_1\omega_1 + K_1\int(\omega_1 - \omega_2)dt = T \qquad .....(3)$$

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + K_2\int\omega_2 dt + K_1\int(\omega_2 - \omega_1)dt = 0 \qquad .....(4)$$

T-V ANALOGOUS CIRCUIT:





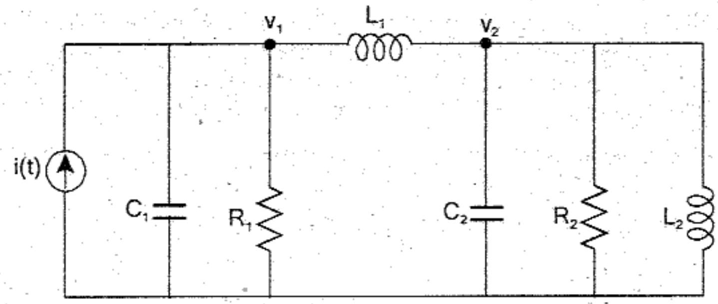
The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig are given below

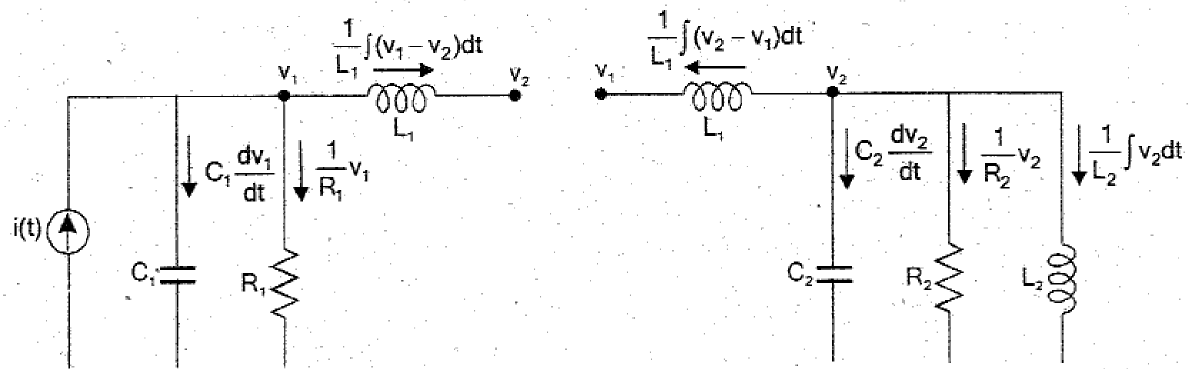
$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int(i_1-i_2)=e(t) \qquad .....(5)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2dt+\frac{1}{C_1}\int(i_2-i_1)dt=0 \qquad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

T-I ANALOGOUS CIRCUIT:





The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$ $\quad B_1 \rightarrow 1/R_1$ $\quad \omega_1 \rightarrow v_1$ $\quad J_1 \rightarrow C_1$ $\quad K_1 \rightarrow 1/L_1$

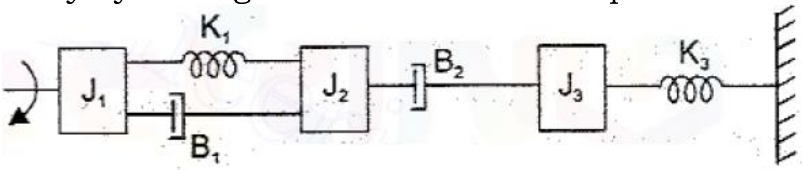
$B_2 \rightarrow 1/R_2$ $\quad \omega_2 \rightarrow v_2$ $\quad J_2 \rightarrow C_2$ $\quad K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig are given below

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \qquad .....(7)$$

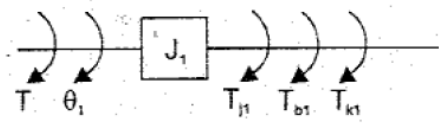
$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{L_1}\int(v_2-v_1)dt=0 \qquad .....(8)$$

4) Write the differential equations governing the mechanical system shown in the fig. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing the mesh and nodal equations.



SOL:

The free body diagram of J_1 is shown in fig The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.



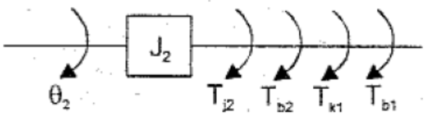
$$T_{j1} = J_1\frac{d^2\theta_1}{dt^2} \; ; \; T_{b1} = B_1\frac{d(\theta_1-\theta_2)}{dt}$$

$$T_{k1} = K_1(\theta_1-\theta_2)$$

By Newton's second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1\frac{d^2\theta_1}{dt^2} + B_1\frac{d(\theta_1-\theta_2)}{dt} + K_1(\theta_1-\theta_2) = T \qquad .....(1)$$

The free body diagram of J_2 is shown in fig The opposing torques are marked as T_{j2}, T_{b2}, T_{b1} and T_{k1}.



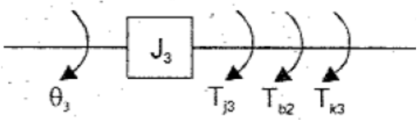
$$T_{j2} = J_2\frac{d^2\theta_2}{dt^2} \; ; \qquad T_{b2} = B_2\frac{d(\theta_2-\theta_3)}{dt}$$

$$T_{k1} = K_1(\theta_2-\theta_1) \; ; \; T_{b1} = B_1\frac{d(\theta_2-\theta_1)}{dt}$$

By Newton's second law, $T_{j2} + T_{b2} + T_{b1} + T_{k1} = 0$

$$J_2\frac{d^2\theta_2}{dt^2} + B_2\frac{d(\theta_2-\theta_3)}{dt} + B_1\frac{d(\theta_2-\theta_1)}{dt} + K_1(\theta_2-\theta_1) = 0 \qquad .....(2)$$

The free body diagram of J_3 is shown in fig The opposing torques are marked as T_{j3}, T_{b2}, and T_{k3}.



$$T_{j3} = J_3\frac{d^2\theta_3}{dt^2} \; ; \; T_{b2} = B_2\frac{d(\theta_3-\theta_2)}{dt} \; ; \; T_{k3} = K_3\theta_3$$

By Newton's second law, $T_{j3} + T_{b2} + T_{k3} = 0$

$$\therefore J_3\frac{d^2\theta_3}{dt^2} + B_2\frac{d(\theta_3-\theta_2)}{dt} + K_3\theta_3 = 0 \qquad .....(3)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \; ; \; \frac{d\theta}{dt} = \omega \;\text{ and }\; \theta = \int \omega \, dt\right)$$

$$J_1\frac{d\omega_1}{dt} + B_1(\omega_1-\omega_2) + K_1\int(\omega_1-\omega_2)\,dt = T \qquad .....(4)$$

$$J_2\frac{d\omega_2}{dt} + B_1(\omega_2-\omega_1) + B_2(\omega_2-\omega_3) + K_1\int(\omega_2-\omega_1)\,dt = 0 \qquad .....(5)$$

$$J_3\frac{d\omega_3}{dt} + B_2(\omega_3-\omega_2) + K_3\int\omega_3 dt = 0 \qquad .....(6)$$

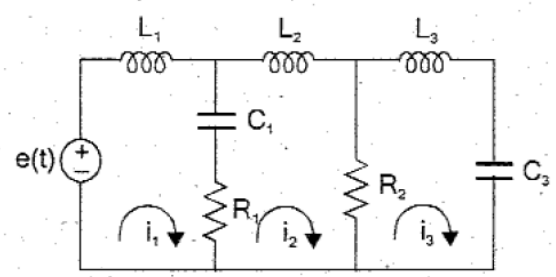
T-V ANALOGOUS CIRCUIT:

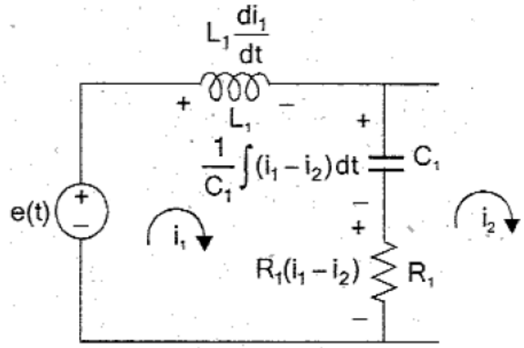
The electrical analogous elements for the elements of mechanical rotational system are given below.

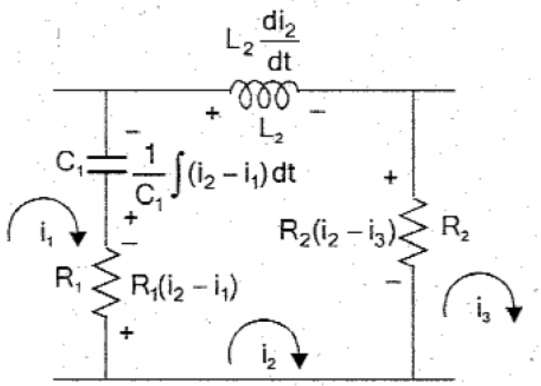
$T \rightarrow e(t)$ $\omega_1 \rightarrow i_1$ $J_1 \rightarrow L_1$ $B_1 \rightarrow R_1$ $K_1 \rightarrow 1/C_1$

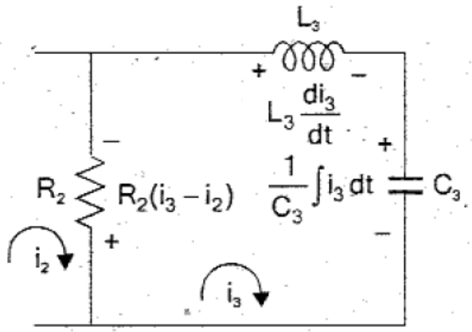
$\omega_2 \rightarrow i_2$ $J_2 \rightarrow L_2$ $B_2 \rightarrow R_2$ $K_3 \rightarrow 1/C_3$

$\omega_3 \rightarrow i_3$ $J_3 \rightarrow L_3$









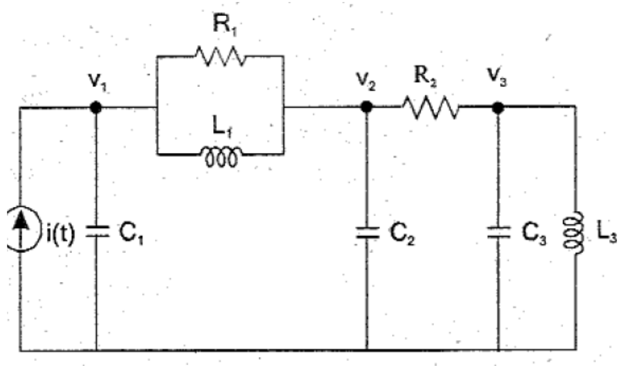
The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig are given below

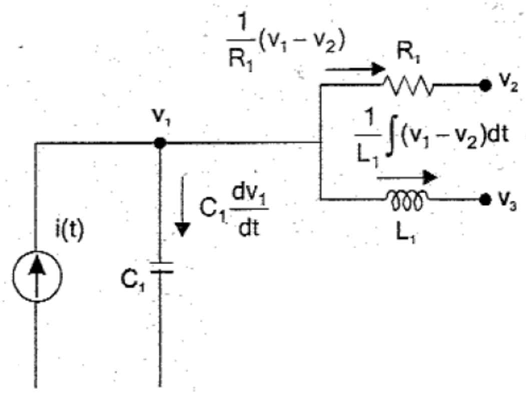
$$L_1\frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int(i_1 - i_2)dt = e(t) \quad .....(7)$$

$$L_2\frac{di_2}{dt} + R_1(i_2 - i_1) + R_2(i_2 - i_3) + \frac{1}{C_1}\int(i_2 - i_1)dt = 0 \quad .....(8)$$

$$L_3\frac{di_3}{dt} + R_2(i_3 - i_2) + \frac{1}{C_3}\int i_3 dt = 0 \quad .....(9)$$

T-I ANALOGOUS CIRCUIT:



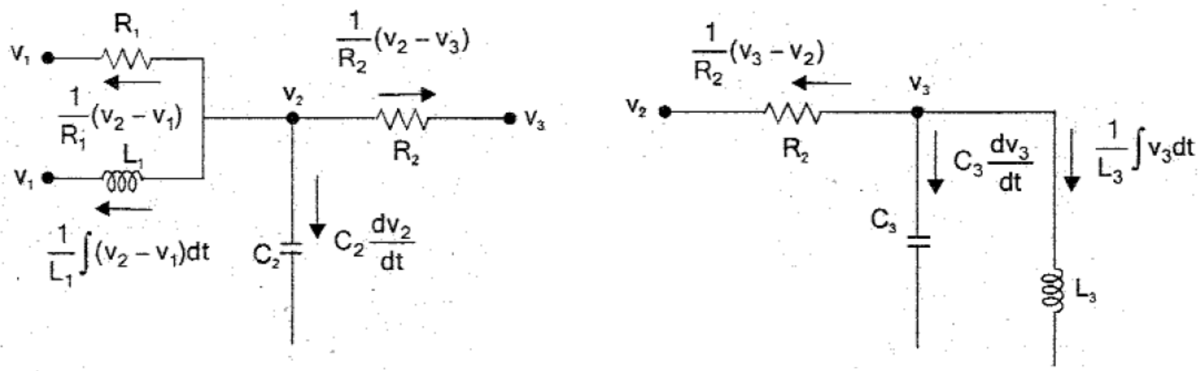


The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$ $\omega_1 \rightarrow v_1$ $J_1 \rightarrow C_1$ $B_1 \rightarrow 1/R_1$ $K_1 \rightarrow 1/L_1$

$\omega_2 \rightarrow v_2$ $J_2 \rightarrow C_2$ $B_2 \rightarrow 1/R_2$ $K_3 \rightarrow 1/L_3$

$\omega_3 \rightarrow v_3$ $J_3 \rightarrow C_3$



The node basis equations using Kirchoff's current law for the circuit shown in fig are given below

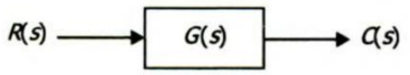
$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \quad .....(10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_1}(v_2-v_1)+\frac{1}{R_2}(v_2-v_3)+\frac{1}{L_1}\int(v_2-v_1)\,dt=0 \quad .....(11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_2}(v_3-v_2)+\frac{1}{L_3}\int v_3\,dt=0 \quad .....(12)$$

BLOCK DIAGRAM REPRESENTATION OF SYSTEMS

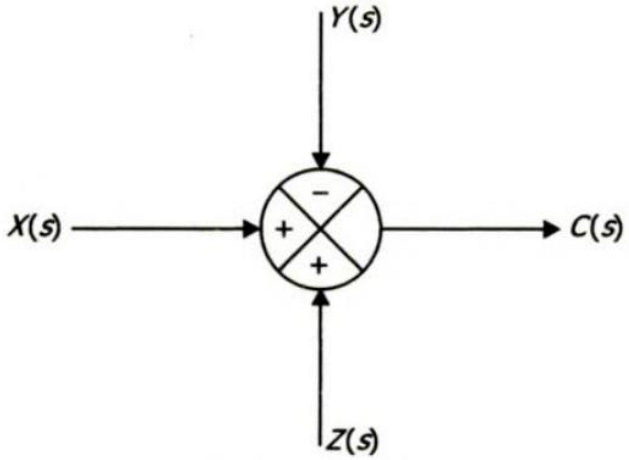
Block diagram: The shorthand pictorial representation of the cause-and-effect relationship between the input and output of a physical system is known as block diagram. Figure shows the representation of a block diagram



Output: The value of input multiplied by the block gain is known as output. From Fig.

$$C(s) = G(s)R(s)$$

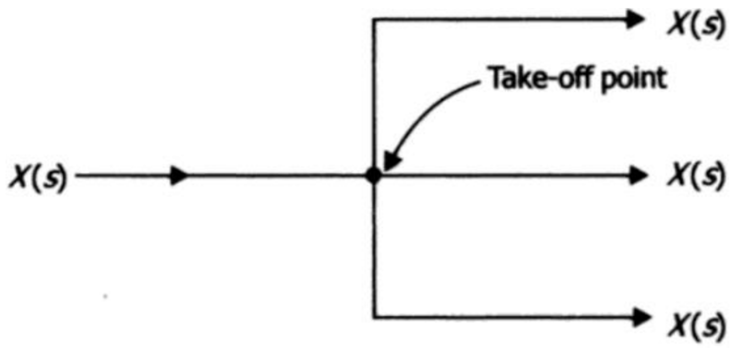
Summing point: At summing point, two or more signals can be added or subtracted.



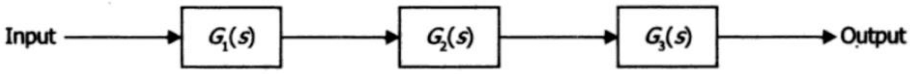
In Fig. $X(s)$, $Y(s)$, and $Z(s)$ are the inputs while $C(s)$ is the output. From Fig. it can be written as

$$C(s) = X(s) - Y(s) + Z(s)$$

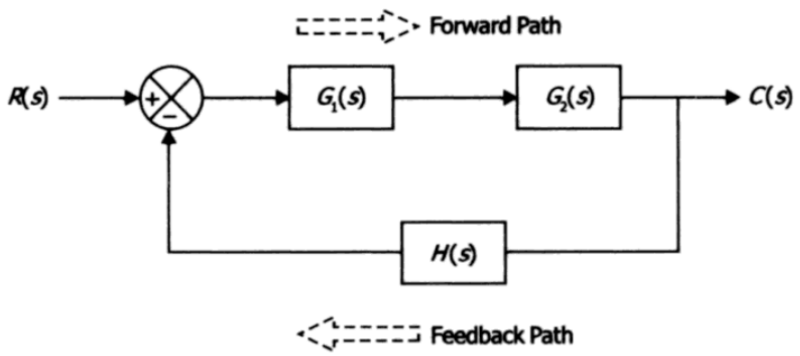
Take off point: The point at which the output signal of any block can be applied to two or more points is known as take-off point. Figure shows a take-off point. The output signal is analogous to voltage not the current.



Forward path: The direction of flow of signal from input to output is known as forward path. Figure shows a forward path. Individual block gains are $G_1(s)$, $G_2(s)$, and $G_3(s)$ in Fig.   from input to output of the system.



Feedback path: The direction of flow of signal from output to input is known as feedback path. Figure shows the feedback path.



Advantages of Block Diagram Representation

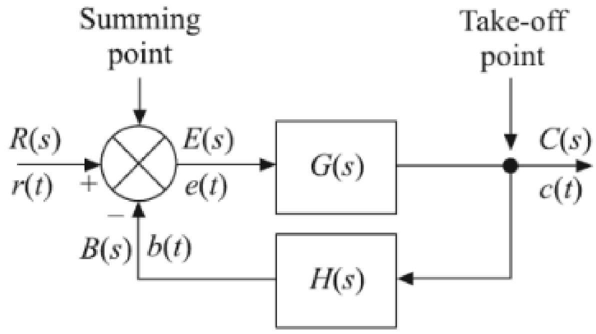
(i) It facilitates easier representation of complex systems.
(ii) Calculation of transfer function by block diagram reduction techniques is easy.
(iii) Performance analysis of a complex system is simplified by determining its transfer function.
(iv) It facilitates easier access of individual elements in a system that is represented by a block diagram.
(v) It facilitates visualization of operation of the whole system by the flow of signals.

Disadvantages of Block Diagram Representation

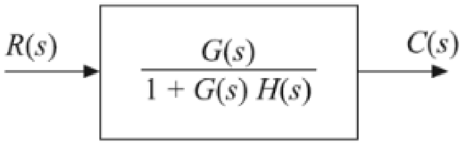
(i) It is difficult to determine the actual composition of individual elements in a system.
(ii) Representation of a system using block diagram is not unique.
(iii) The main source of signal flow cannot be represented definitely in a block diagram.

BLOCK DIAGRAM OF A CLOSED-LOOP SYSTEM

Figure   shows an example of a block diagram of a closed-loop system. The output $C(s)$ is fed back to the summing point where it is compared with the reference input. When the output is fed back to the summing point for comparison with the input, it is necessary to convert the form of the output signal to that of the input signal. The role of the feedback element is to modify the output before it is compared with the input. This conversion is accomplished by the feedback element whose transfer function is $H(s)$. The output of the block $C(s)$ in this case is obtained by multiplying the transfer function $G(s)$ by the input to the block $E(s)$. The feedback signal that is fed back to the summing point for comparison with the input is $B(s) = C(s)H(s)$.



(a)



(b)

The following terminology is defined with reference to the diagram of Figure

$r(t)$, $R(s)$ = reference input (command)
$c(t)$, $C(s)$ = output (controlled variable)
$b(t)$, $B(s)$ = feedback signal
$e(t)$, $E(s)$ = error signal
$G(s)$ = forward path transfer function
$H(s)$ = feedback path transfer function

$G(s)\ H(s) = L(s)$ = loop transfer function or open-loop transfer function

$T(s) = \dfrac{C(s)}{R(s)}$ = closed-loop transfer function or system transfer function

The closed-loop transfer function $T(s)$ can be expressed as a function of $G(s)$ and $H(s)$. From Figure

$$C(s) = G(s)\ E(s)$$

and

$$B(s) = C(s)\ H(s)$$

The error signal is

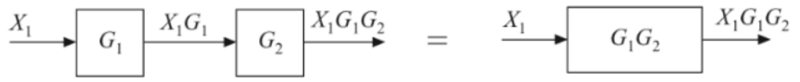
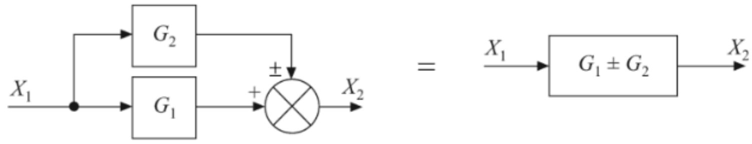
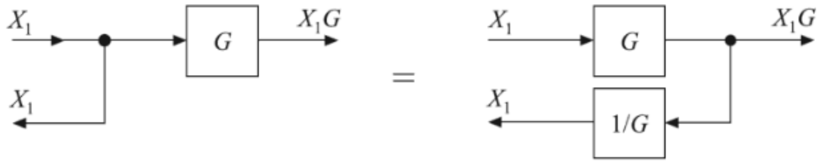
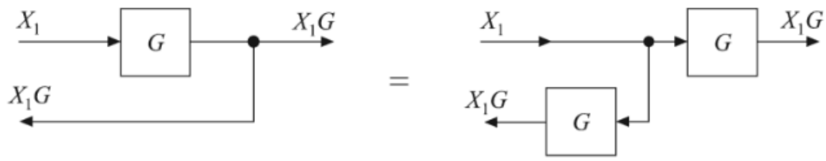
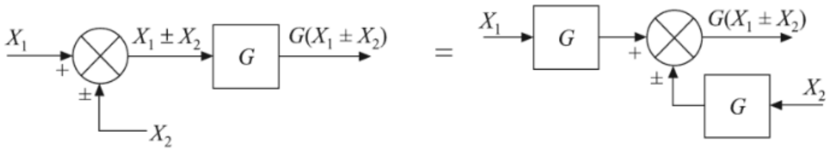
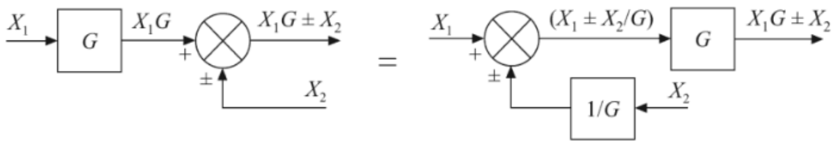
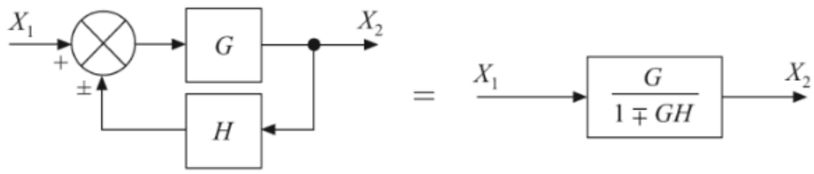
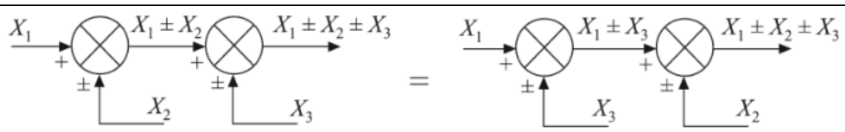
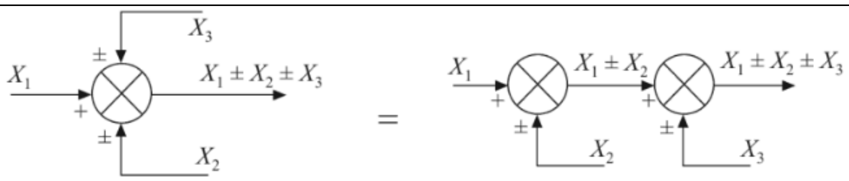
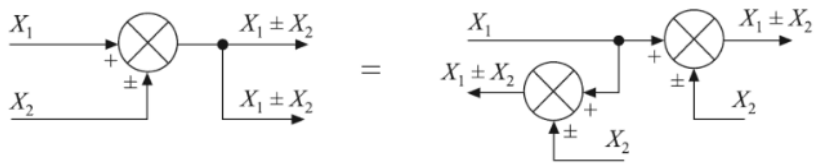
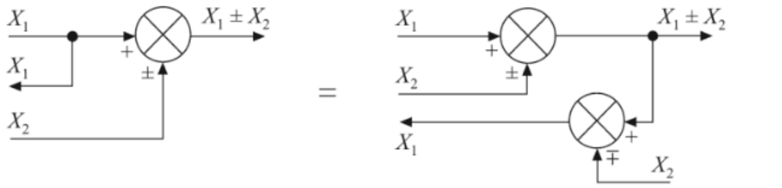
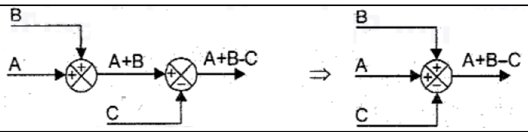
$$E(s) = R(s) - B(s)$$

Substituting Eqs.

$$C(s) = G(s)[R(s) - B(s)]$$
$$= G(s)R(s) - G(s)B(s)$$
$$= G(s)R(s) - G(s)H(s)C(s)$$
$$C(s)[1 + G(s)H(s)] = G(s)R(s)$$
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

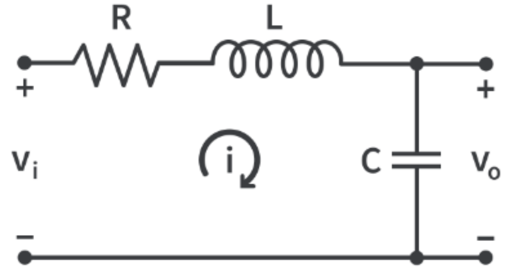
Use **+ sign for negative feedback** and Use **– sign for positive feedback.**

BLOCK DIAGRAM ALGEBRA - RULES FOR REDUCTION OF BLOCK DIAGRAM

RULE-1	Combining the blocks in Cascade	
RULE-2	Combining the blocks in Parallel	
RULE-3	Moving a take-off point after the block	
RULE-4	Moving a take-off point ahead of a block	
RULE-5	Moving a summing point after the block	
RULE-6	Moving a summing point ahead of the block	
RULE-7	Elimination of feedback loop	
RULE-8	Interchanging the summing points	
RULE-9	Splitting a summing point	
RULE-10	Moving a take-off point ahead of a summing point	
RULE-11	Moving a take-off point after a summing point	
RULE-12	Combining the summing points	

PROBLEMS

1) Draw the block diagram of series RLC circuit as shown in the fig.



SOL:

Applying Kirchhoff's voltage law to the loop shown above,

$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + v_o(t)$$

$$v_o(t) = \frac{1}{C}\int i(t)dt$$

Laplace transformation of the above equations with initial conditions assumed zero will be:

$$V_i(s) = RI(s) + sLI(s) + V_o(s)$$

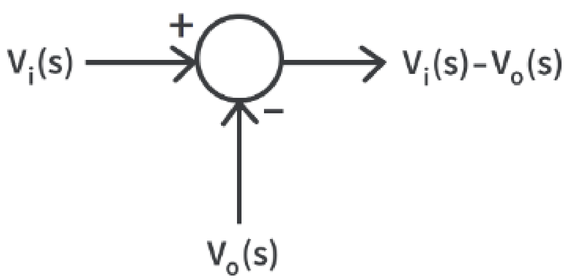
$$I(s)(R + sL) = V_i(s) - V_o(s)$$

$$I(s) = \frac{V_i(s) - V_o(s)}{R + sL}$$

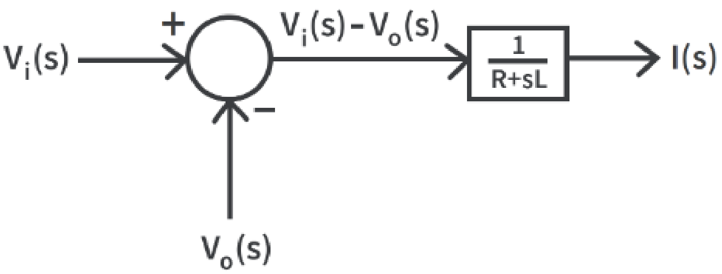
$$V_o(s) = \frac{1}{sC}I(s)$$

$$I(s) = [V_i(s) - V_o(s)]\,\frac{1}{R + sL}$$

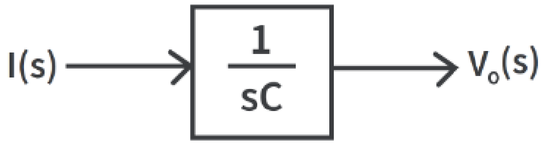
First we shall use a summing point.



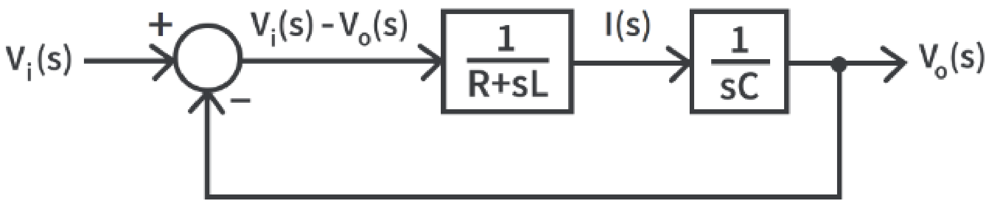
The output of the summing point is passed through a block of transfer function: $\dfrac{1}{R+sL}$



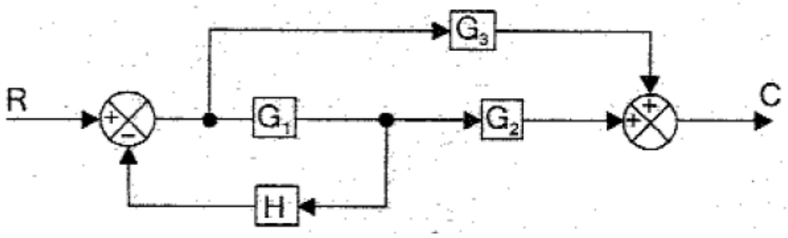
Next, we shall use the other equation, $V_o(s) = \dfrac{1}{sC} I(s)$



We combine the above two blocks and then with the help of a take off point, we connect the output to the summing point where we need the output variable as one of the inputs.

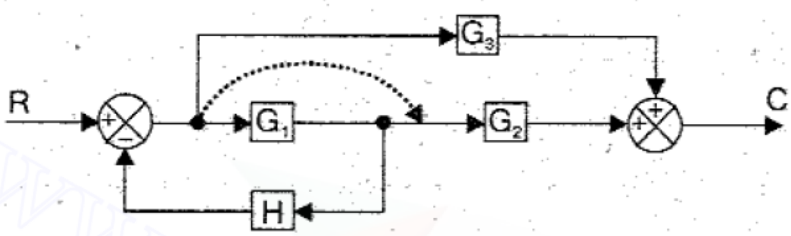


2) Reduce the block diagram shown in the fig. and find C / R.

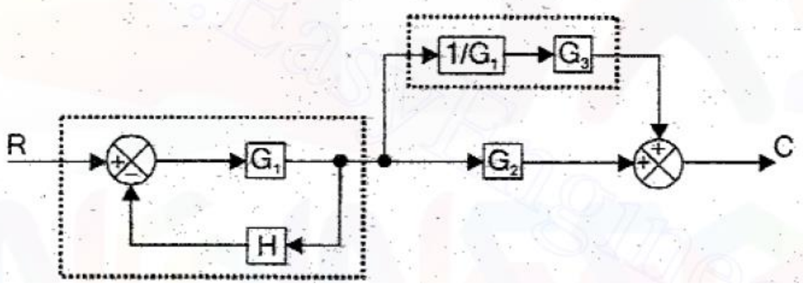


SOL:

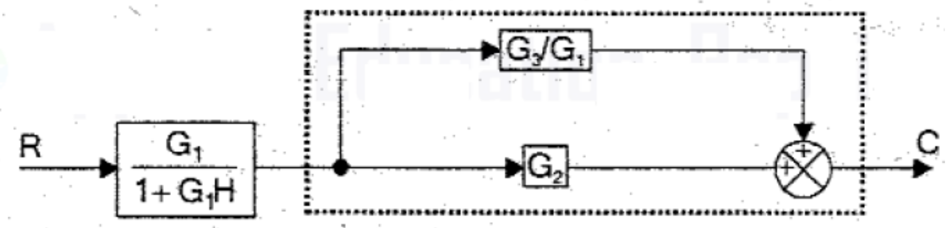
Step 1: Move the branch point after the block.



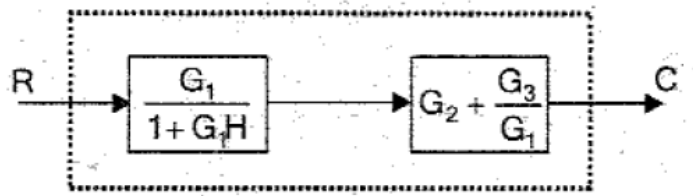
Step 2: Eliminate the feedback path and combining blocks in cascade.



Step 3: Combining parallel blocks

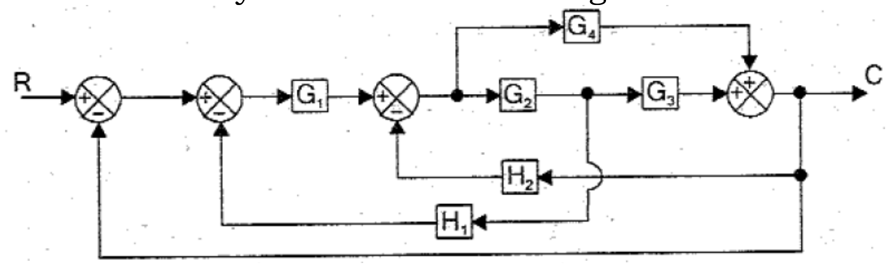


Step 4: Combining blocks in cascade



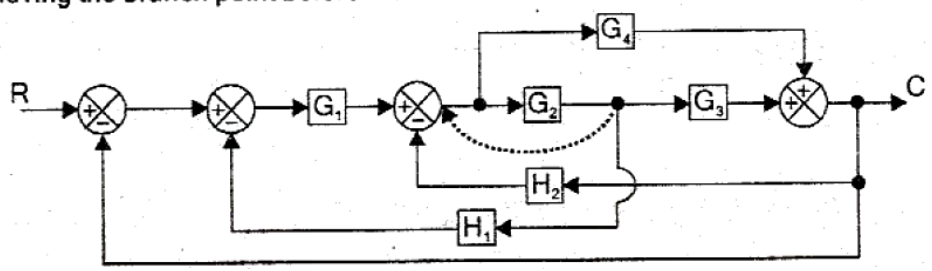
$$\frac{C}{R}=\left(\frac{G_1}{1+G_1H}\right)\left(G_2+\frac{G_3}{G_1}\right)=\left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2+G_3}{G_1}\right)=\frac{G_1G_2+G_3}{1+G_1H}$$

3) Using the block diagram reduction techniques, find the closed loop transfer function of the system whose block diagram is shown in the fig.

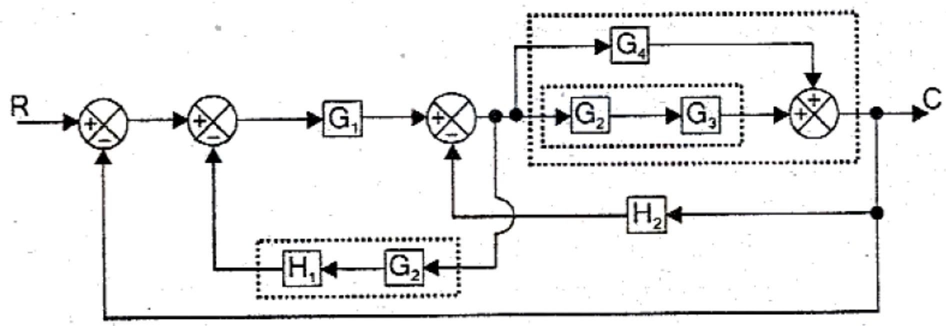


SOL:

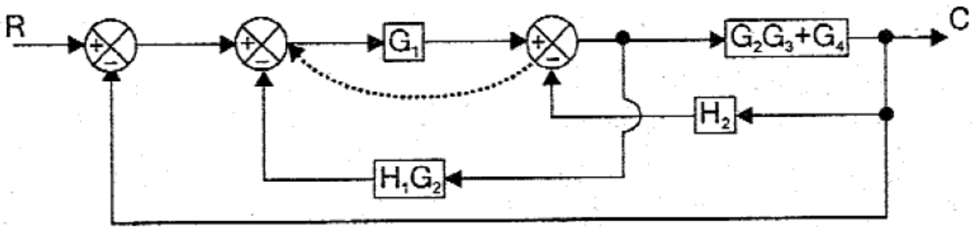
Step 1: Moving the branch point before the block



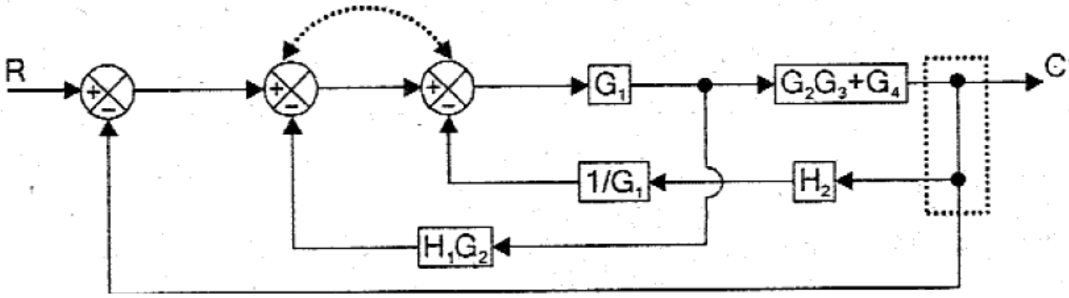
Step 2: Combining the blocks in cascade and eliminating parallel blocks



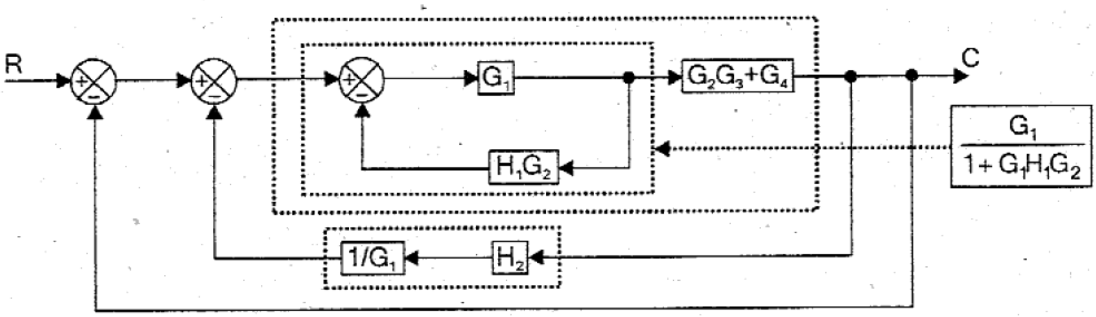
Step 3: Moving summing point before the block.



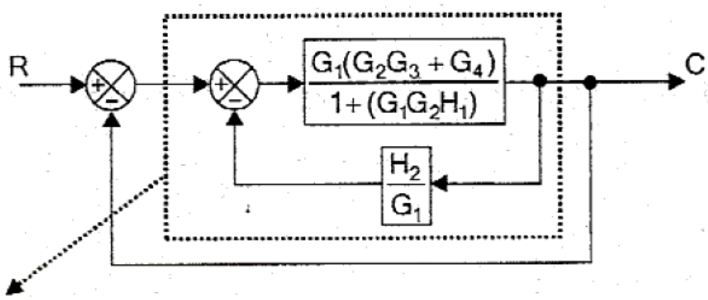
Step 4: Interchanging summing points and modifying branch points.



Step 5: Eliminating the feedback path and combining blocks in cascade

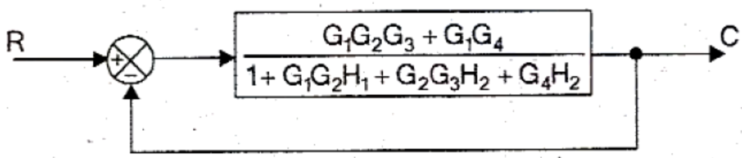


Step 6: Eliminating the feedback path



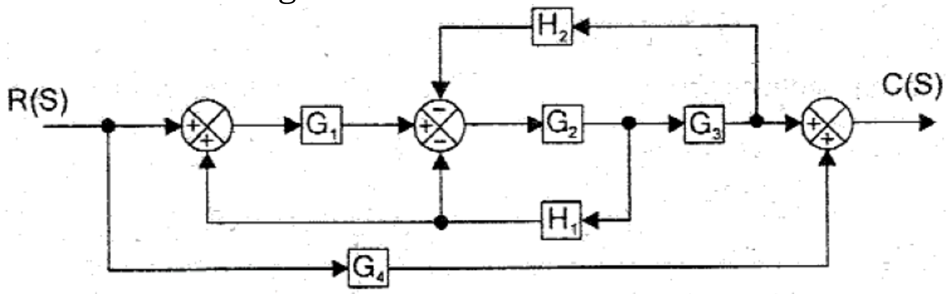
$$\frac{\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}\dfrac{H_2}{G_1}} \Rightarrow \frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1}}{\dfrac{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}{1+G_1G_2H_1}} \Rightarrow \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}$$

Step 7: Eliminating the feedback path



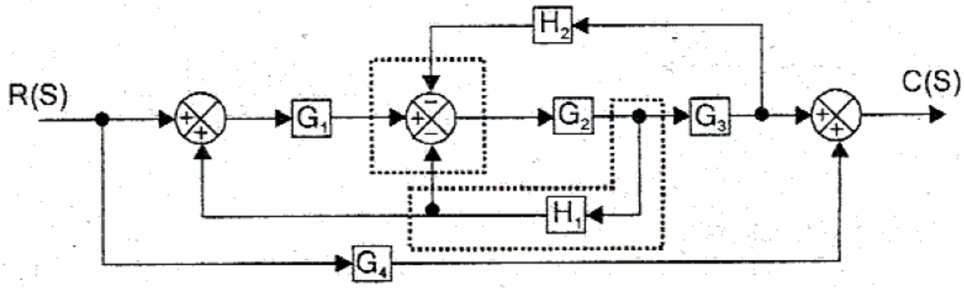
$$\frac{C}{R}=\frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}} = \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

4) Obtain the transfer function C(s) / R(s) of the system whose block diagram is shown in the fig.

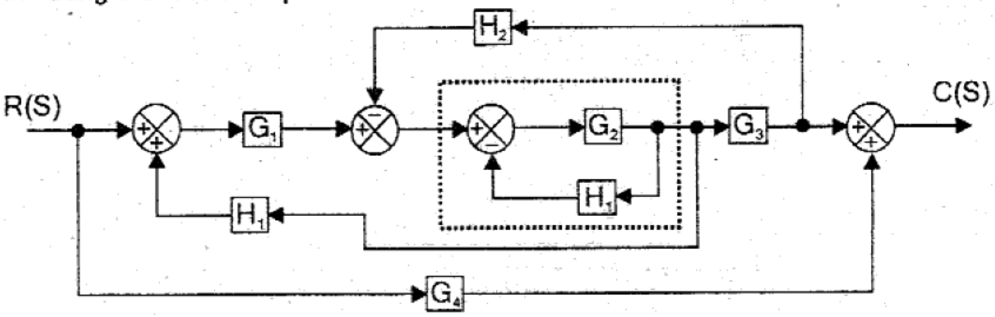


SOL:

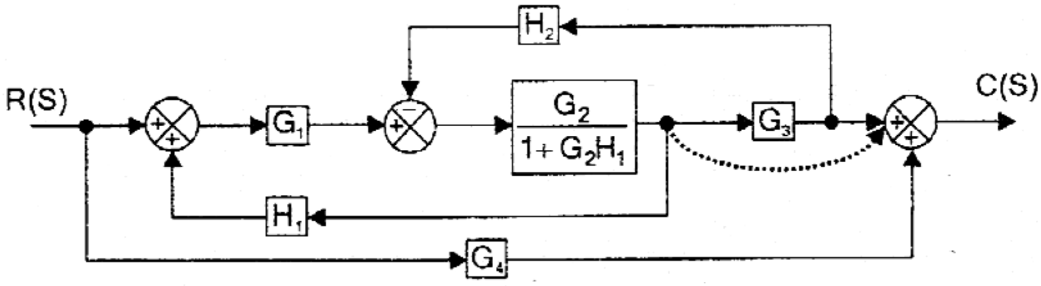
Step 1 : Splitting the summing point and rearranging the branch points



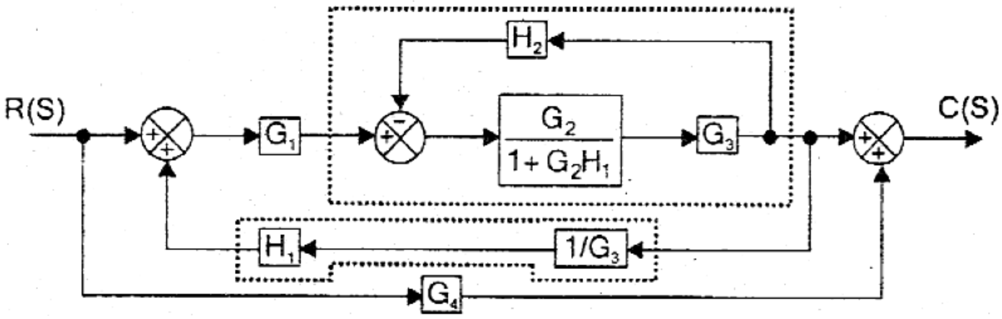
Step 2 : Eliminating the feedback path



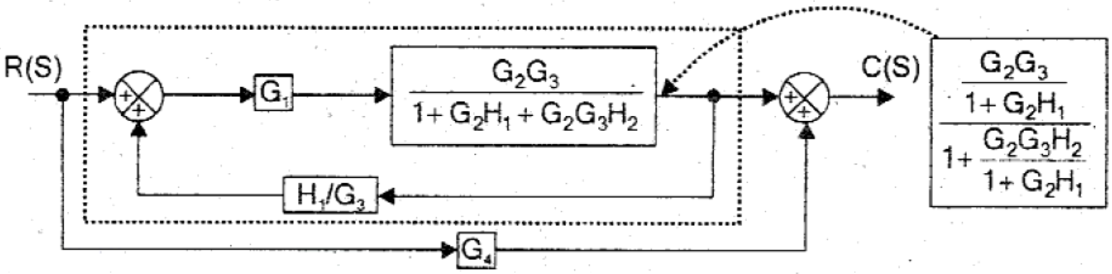
Step 3 : Shifting the branch point after the block.



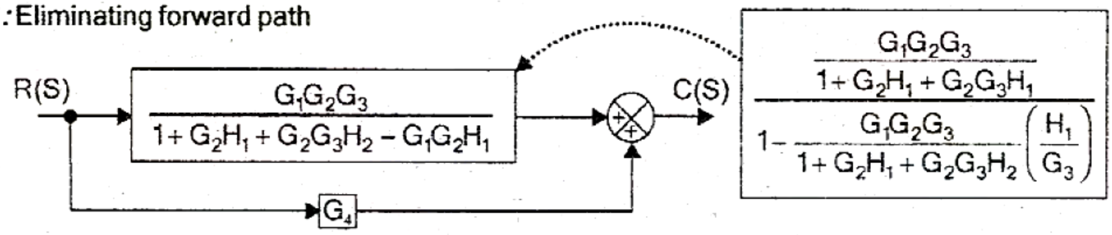
Step 4 : Combining the blocks in cascade and eliminating feedback path



Step 5 : Combining the blocks in cascade and eliminating feedback path

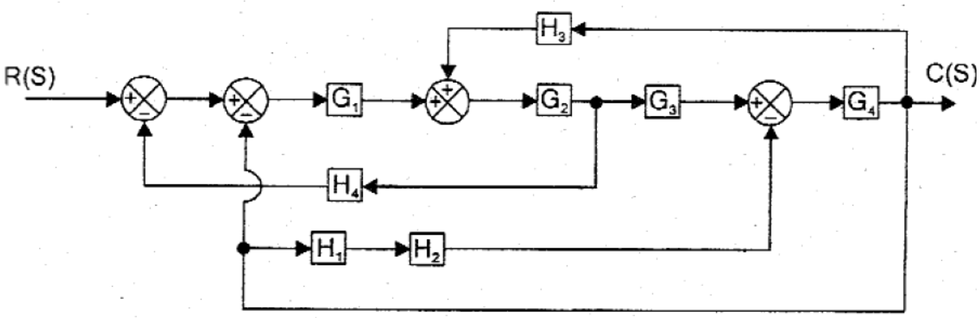


Step 6 : Eliminating forward path



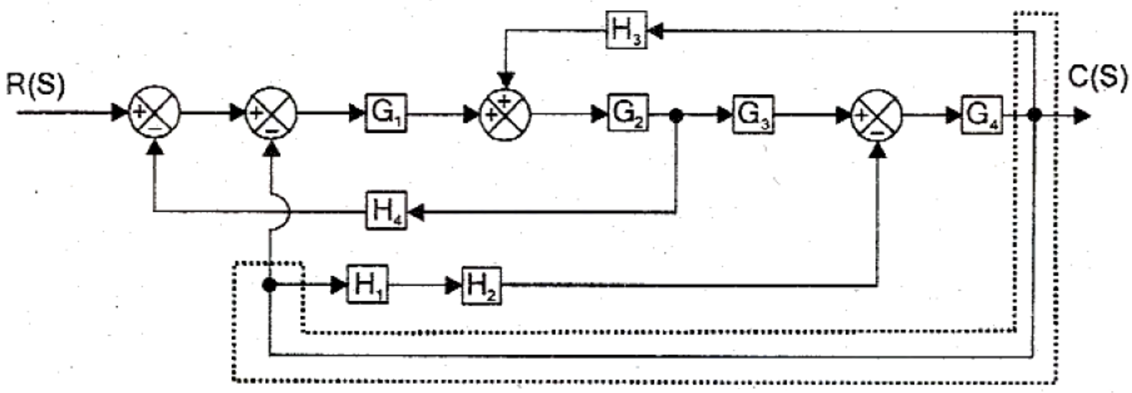
$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$$

5) Using the block diagram reduction techniques, find the closed loop transfer function C(s)/R(s) of the system whose block diagram is shown in the fig.

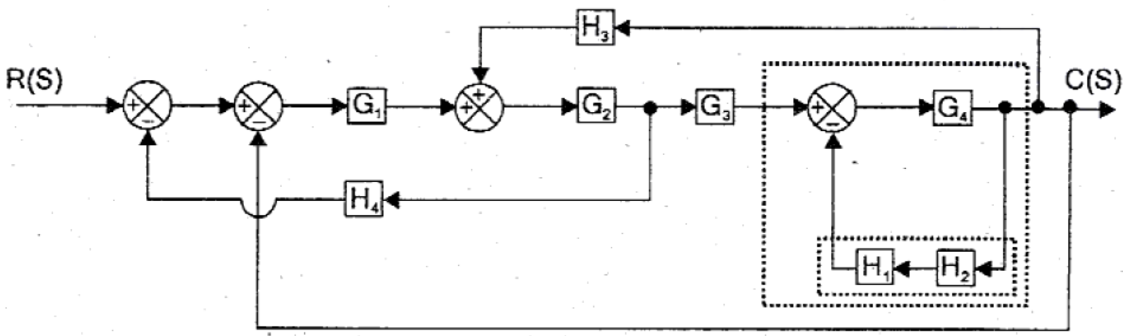


SOL:

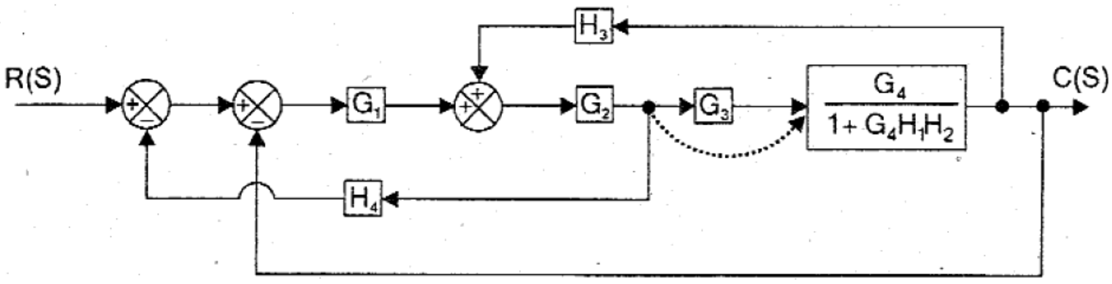
Step 1 : Rearranging the branch points



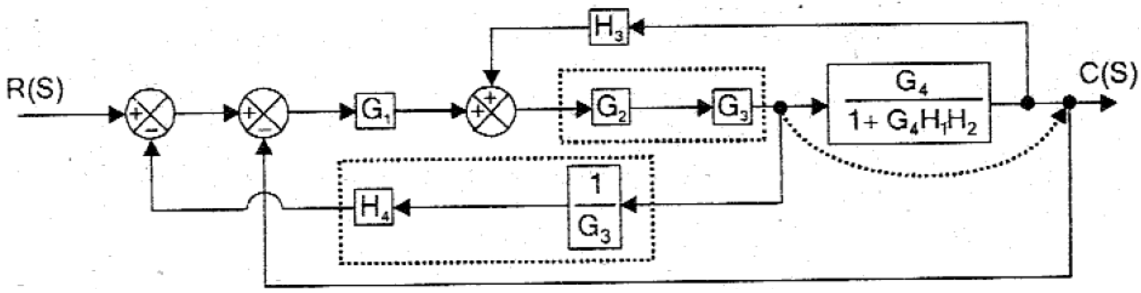
Step 2 : Combining the blocks in cascade and eliminating the feedback path.



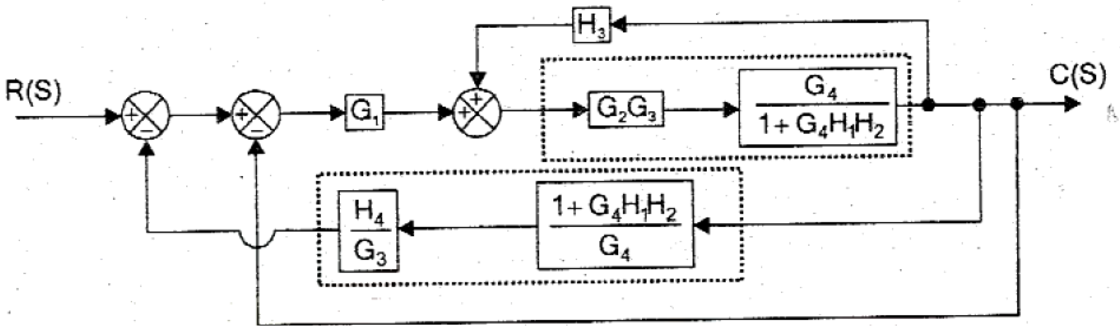
Step 3 : Moving the branch point after the block.



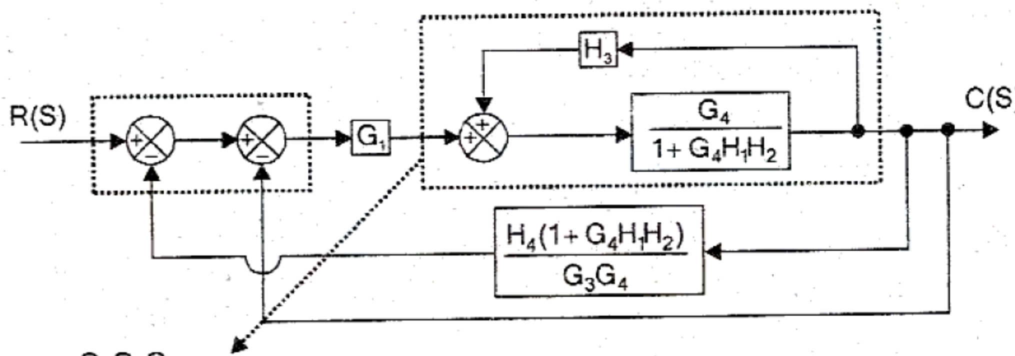
Step 4 : Moving the branch point and combining the blocks in cascade.



Step 5 : Combining the blocks in cascade

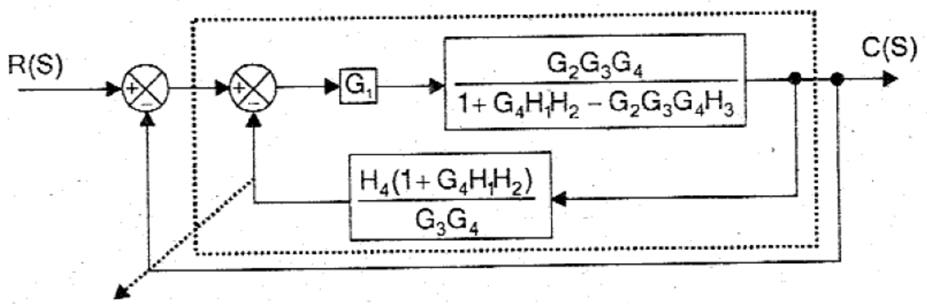


Step 6 : Eliminating feedback path and interchanging the summing points.



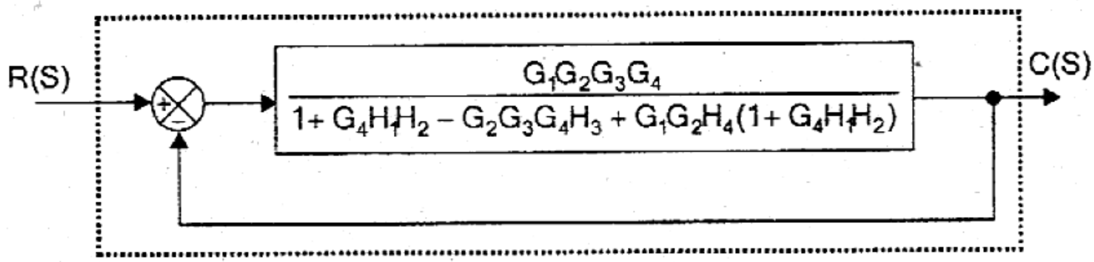
$$\frac{\dfrac{G_2G_3G_4}{1+G_4H_1H_2}}{1-\dfrac{G_2G_3G_4H_3}{1+G_4H_1H_2}} = \frac{G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}$$

Step 7 : Combining the blocks in cascade and eliminating the feedback path



$$\frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}}{1+\left(\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}\right)\left(\dfrac{H_4(1+G_4H_1H_2)}{G_3G_4}\right)}=\frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}$$

Step 8 : Eliminating the unity feedback path.



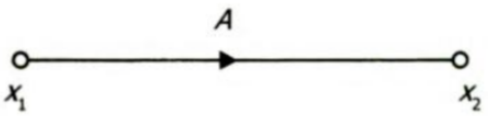
$$\therefore \frac{C(s)}{R(s)}=\frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}{1+\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}$$

$$=\frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)+G_1G_2G_3G_4}$$

$$=\frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

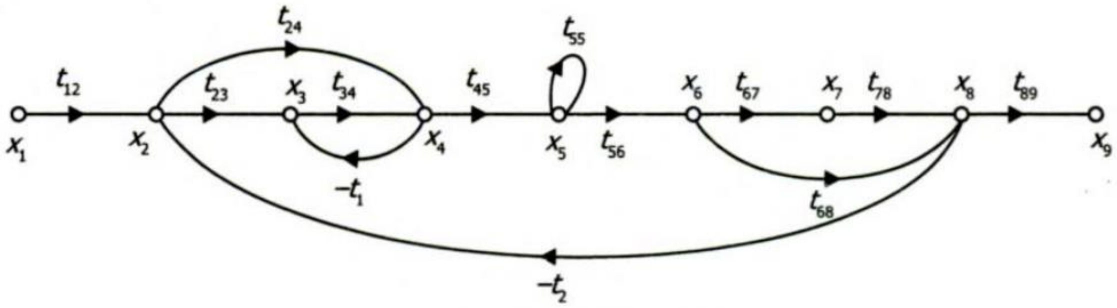
SIGNAL FLOW GRAPH (SFG)

If the set of equations describing a system is known, it is possible to represent the system in another pictorial form which is known as signal-flow graph. It may be regarded as a simplified version of a block diagram. Signal-flow graph is a pictorial representation of a system which displays graphically the transmission of signals in it. S.J. Mason introduced the signal-flow graph for representing the cause and effect of linear systems which are modelled by algebraic equations. Although there is difference between physical appearances of the signal-flow graph (SFG) and block diagram, the signal-flow graph is constrained by more rigid mathematical rules whereas the usage of block diagram's notation is more liberal. In a signal-flow graph, all dependent and independent variables are represented by the nodes, and the lines joining the nodes are known as branches. A branch is associated with transmission gain and an arrow. Figure shows the pictorial representation of a system having two variables where $x_2 = Ax_1$.



BASIC DEFINITIONS IN SFG

Consider the following signal flow graph.



Input or *source node:* The node having only outgoing branches is called input or source node. In Fig. x_1 is the input or source node.

Sink node: The node having only incoming branches is called sink node or output node. In Fig. x_9 is the sink or output node.

Chain node: The node having both incoming and outgoing branches is know as chain node. In Fig. $x_2, x_3, x_4, x_5, x_6, x_7$, and x_8 are all chain nodes.

Forward path: A path from input to output is known as forward path.

In Figure there are four forward paths as follows:

$x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9$: First forward path

$x_1 - x_2 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9$: Second forward path

$x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_8 - x_9$: Third forward path

$x_1 - x_2 - x_4 - x_5 - x_6 - x_8 - x_9$: Fourth forward path

In determining forward path, any node should not be traded twice.

Feedback Loop/ Feedback Path: If a loop originates and terminates at the same node, it is known as feedback loop. In Fig. there are three feedback loops or feedback paths as follows:

$x_3 - x_4 - x_3$: First feedback loop

$x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_2$: Second feedback loop, $x_2 - x_1 - x_5 - x_6 - x_8 - x_2$: Third feedback loop.

Self-Loop: A loop that consists of only one node is known as self-loop. In determining forward path or feedback path, the self-loop should not be taken into account. In Fig. t_{55}, at x_5, is the self-loop.

Path Gain: The product of gains going through a forward path is called path gain. In Fig. the path gain for the first forward path is

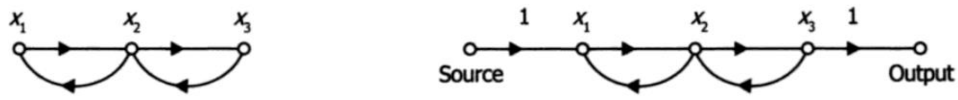
$P_1 = t_{12}t_{23}t_{34}t_{45}t_{56}t_{67}t_{78}t_{89}$

Similarly, the path gains for the second, third and fourth paths are

$P_2 = t_{12}t_{24}t_{45}t_{56}t_{67}t_{78}t_{89}$,

$P_3 = t_{12}t_{23}t_{34}t_{45}t_{56}t_{68}t_{89}$ and $P_4 = t_{12}t_{24}t_{45}t_{56}t_{68}t_{89}$ respectively.

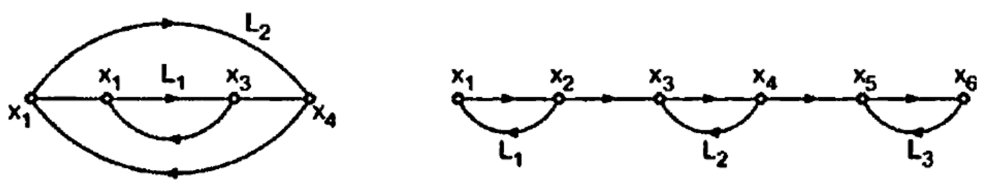
Dummy Node: If the incoming as well as outgoing branches exist at the first and the last node representing input and output variables, these nodes cannot be taken as input and output nodes. In such cases, separate input and output nodes are created by adding branches with gain 1. These nodes are know as dummy nodes. Fig. shows the dummy nodes.



(a) Without input and output nodes (b) Dummy nodes

Non touching loops : If there is no node common in between the two or more loops, such loops are said to be non touching loops.

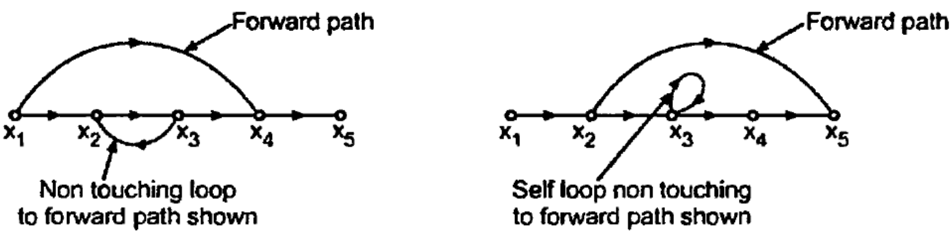
The Fig. (a)&(b) show a combination of non touching loops of two and three loops.



(a) Two non touching loops **(b) Three non touching loops**

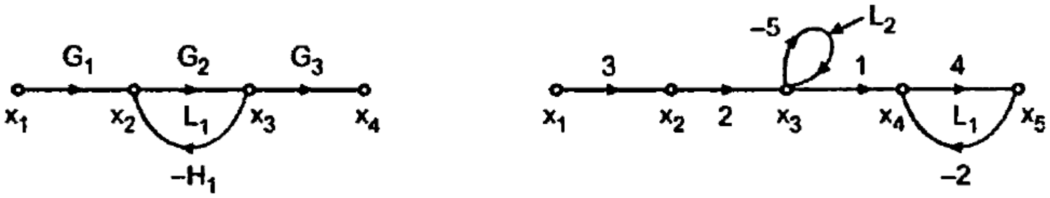
Similarly if there is no node common in between a forward path and a feedback loop, a loop is said to be non touching to that forward path.

The Fig. (a) and (b) shows such a loop which is non touching to a forward path.



(a) **(b)**

Loop Gain : The product of all the gains of the branches forming a loop is called loop gain. For a self loop, gain indicated along it is its gain. Generally such loop gains are denoted by 'L' e.g. L_1, L_2 etc.



(a) **(b)**

In the Fig. (a), there is one loop with gain $L_1 = G_2 \times -H_1 = -G_2H_1$

In the Fig. (b), there are two loops with gains.

$L_1 = 4 \times -2 = -8$ and other self loop with $L_2 = -5$

PROPERTIES OF SIGNAL FLOW GRAPH

The basic properties of signal flow graph are the following :

(i) The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.

(ii) Signal flow graph is applicable to linear systems only.

(iii) A node in the signal flow graph represents the variable or signal.

(iv) A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.

(v) A mixed node which has both incoming and outgoing signals can be treated as an output node by adding an outgoing branch of unity transmittance.

(vi) A branch indicates functional dependence of one signal on the other.

(vii) The signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

(viii) The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

MASON'S GAIN FORMULA – REDUCTION OF SIGNAL FLOW GRAPH

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let, R(s) = Input to the system

C(s) = Output of the system

Now, Transfer function of the system, $T(s) = \frac{C(s)}{R(s)}$

Mason's gain formula states the overall gain of the system [transfer function] as follows,

$$\text{Overall gain, } T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

where, T = $T(s)$ = Transfer function of the system

P_K = Forward path gain of K^{th} forward path

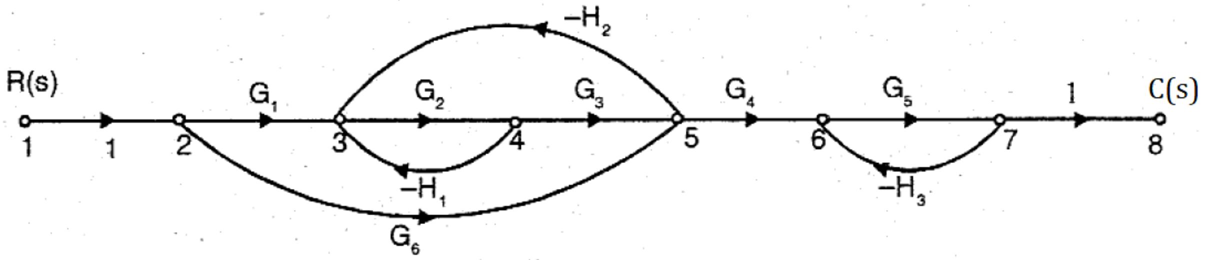
K = Number of forward paths in the signal flow graph

$$\Delta = 1 - (\text{Sum of individual loop gains}) + \begin{pmatrix}\text{Sum of gain products of all possible} \\ \text{combinations of two non-touching loops}\end{pmatrix} - \begin{pmatrix}\text{Sum of gain products of all possible} \\ \text{combinations of three non-touching loops}\end{pmatrix} + \ldots\ldots$$

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path

PROBLEMS

1) Find the overall transfer function of the system whose signal flow graph is shown in the fig.

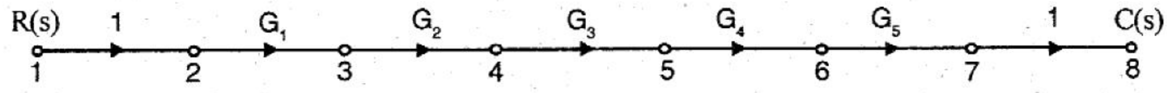


SOL:

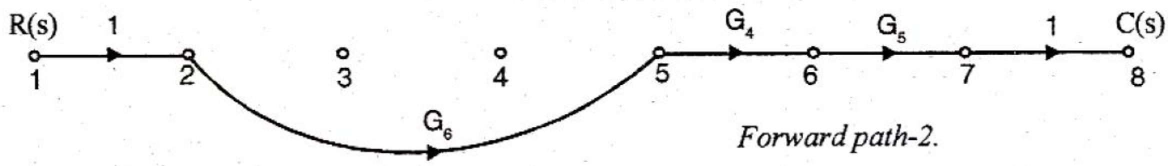
Forward Path Gains

There are two forward paths. $\therefore K = 2$

Let forward path gains be P_1 and P_2.



Forward path-1.



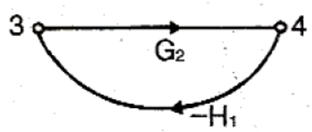
Forward path-2.

Gain of forward path-1, $P_1 = G_1\,G_2\,G_3\,G_4\,G_5$

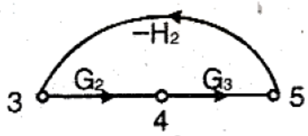
Gain of forward path-2, $P_2 = G_4 G_5 G_6$

Individual Loop Gain

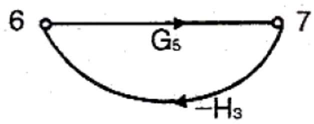
There are three individual loops. Let individual loop gains be P_{11}, P_{21} and P_{31}.



Loop-1.



Loop-2.



Loop-3.

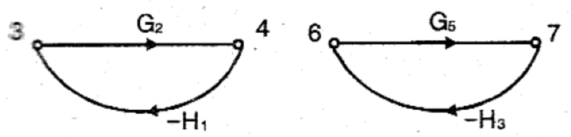
Loop gain of individual loop-1, $P_{11} = -G_2 H_1$

Loop gain of individual loop-2, $P_{21} = -G_2 G_3 H_2$

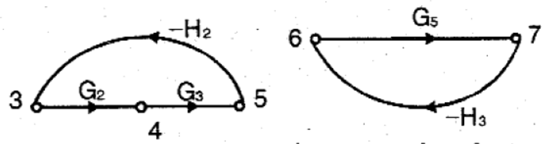
Loop gain of individual loop-3, $P_{31} = -G_5 H_3$

Gain Products of Two Non-touching Loops

There are two combinations of two non-touching loops. Let the gain products of two non touching loops be P_{12} and P_{22}.



First combination of 2 non-touching loops.



Second combination of 2 non-touching loops.

Gain product of first combination of two non touching loops } $P_{12} = P_{11}P_{31} = (-G_2H_1)\,(-G_5H_3) = G_2G_5H_1H_3$

Gain product of second combination of two non touching loops } $P_{22} = P_{21}P_{31} = (-G_2G_3H_2)\,(-G_5H_3) = G_2G_3G_5H_2H_3$

Calculation of Δ and Δ_K

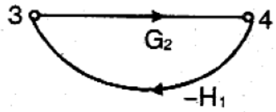
$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$

$= 1 - (-G_2H_1 - G_2G_3H_2 - G_5H_3) + (G_2G_5H_1H_3 + G_2G_3G_5H_2H_3)$

$= 1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3$

$\Delta_1 = 1$, Since there is no part of graph which is not touching with first forward path.

The part of the graph which is non touching with second forward path is shown in fig



$\Delta_2 = 1 - P_{11} = 1 - (-G_2H_1) = 1 + G_2H_1$

Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

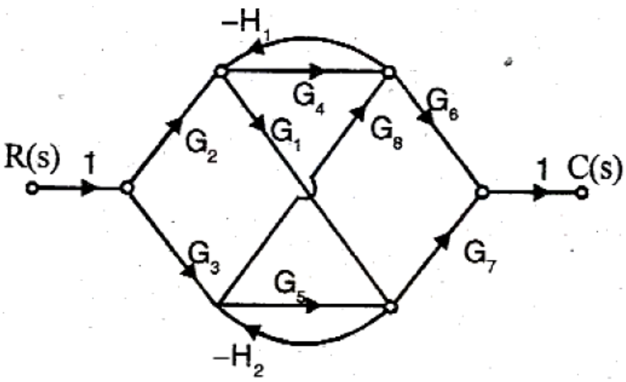
$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is 2 and so K = 2)}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6\,(1 + G_2H_1)}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

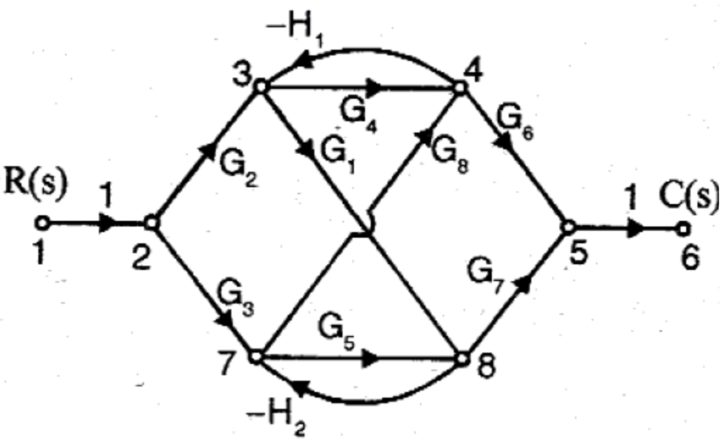
$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6 + G_2G_4G_5G_6H_1}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

$$= \frac{G_2G_4G_5\left[G_1G_3 + G_6 / G_2 + G_6H_1\right]}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

2) Find the overall transfer function of the system whose signal flow graph is shown in the fig.



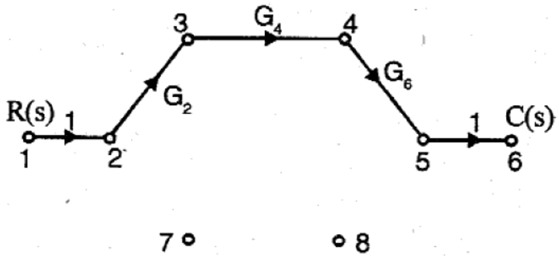
SOL:



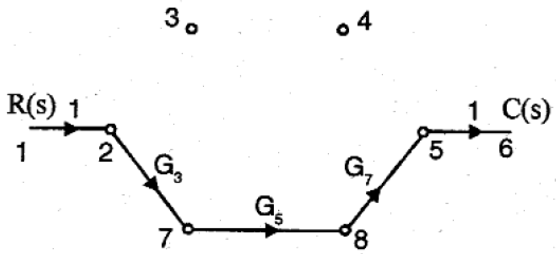
Forward Path Gains

There are six forward paths. $\therefore K = 6$

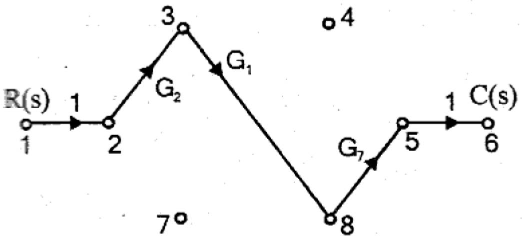
Let the forward path gains be P_1, P_2, P_3, P_4, P_5 and P_6.



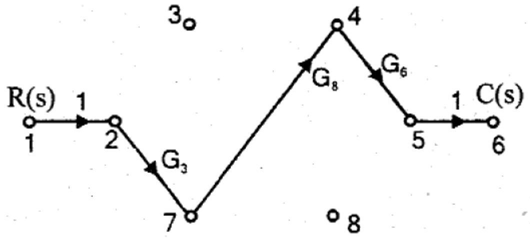
Forward path-1.



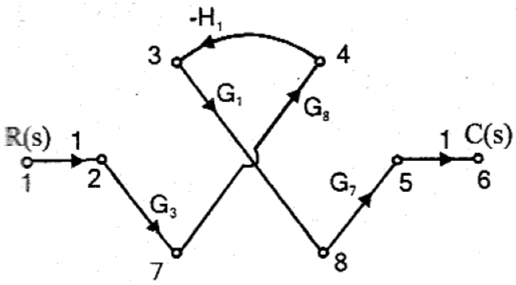
Forward path-2.



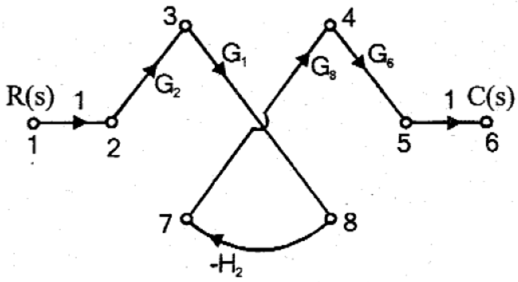
Forward path-3



Forward path-4



Forward path-5



Forward path-6

Gain of forward path-1, $P_1 = G_2 G_4 G_6$

Gain of forward path-2, $P_2 = G_3 G_5 G_7$

Gain of forward path-3, $P_3 = G_1 G_2 G_7$

Gain of forward path-4, $P_4 = G_3 G_8 G_6$

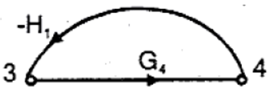
Gain of forward path-5, $P_5 = -G_1G_3G_7G_8H_1$

Gain of forward path-6, $P_6 = -G_1G_2G_6G_8H_2$

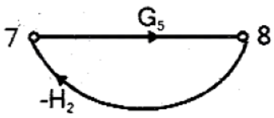
Individual Loop Gain

There are three individual loops.

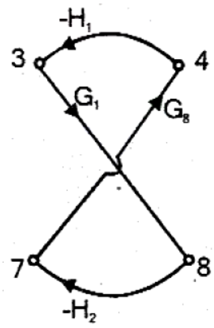
Let individual loop gains be P_{11}, P_{21} and P_{31}.



Loop-1



Loop-2



Loop-3

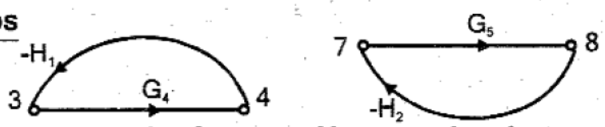
Loop gain of individual loop-1, $P_{11} = -G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_5H_2$

Loop gain of individual loop-3, $P_{31} = G_1G_8H_1H_2$

Gain Products of Two Non-touching Loops

There is only one combination of two non-touching loops Let gain product of two non-touching loops be P_{12}.



Combination of 2 non-touching loops

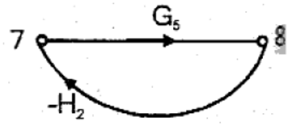
$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non - touching loops}\end{array}\right\} P_{12} = P_{11}P_{21} = (-G_4H_1)\,(-G_5H_2) = G_4G_5H_1H_2$$

Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 - (-G_4H_1 - G_5H_2 + G_1G_8H_1H_2) + G_4G_5H_1H_2$$

$$= 1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2$$

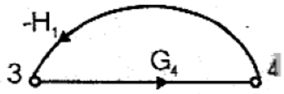
The part of the graph non-touching forward path - 1 is shown in fig



$\therefore \Delta_1 = 1 - (-G_5H_2) = 1 + G_5H_2$

The part of the graph non-touching forward path -2 is shown in fig

$\therefore \Delta_2 = 1 - (-G_4H_1) = 1 + G_4H_1$



There is no part of the graph which is non-touching with forward paths 3, 4, 5 and 6.

$\therefore \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$

Transfer Function, T

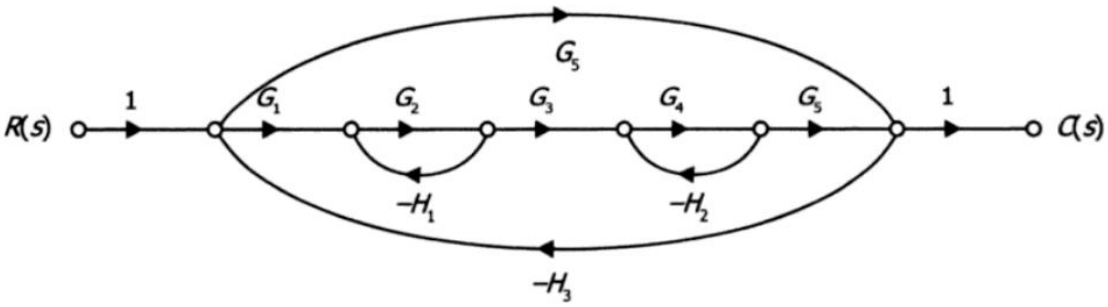
By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\left(\sum_K P_K\,\Delta_K\right) \qquad \text{(Number of forward paths is six and so K = 6)}$$

$$= \frac{1}{\Delta}\,(P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6)$$

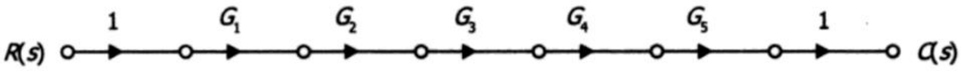
$$= \frac{G_2G_4G_6(1+G_5H_2) + G_3G_5G_7(1+G_4H_1) + G_1G_2G_7 + G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8H_2}{1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2}$$

3) Find the overall transfer function C(s)/R(s) of the system whose signal flow graph is shown in the fig.



SOL:

Step 1: There are two forward paths as follows:



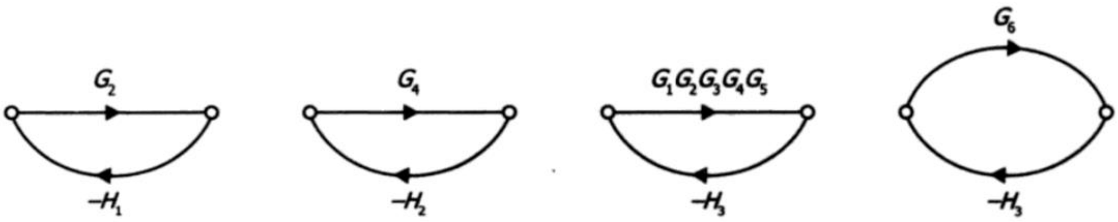
(a) First forward path

(b) Second forward path

Gain of first forward path $(P_1) = G_1G_2G_3G_4G_5$

Gain of second forward path $(P_2) = G_6$

Step 2: There are four loops as shown in Fig.



Loop gain (L_1) of the loop shown in Fig. $= -G_2H_1$

Loop gain (L_2) of the loop shown in Fig. $= -G_4H_2$

Loop gain (L_3) of the loop shown in Fig. $= -G_1G_2G_3G_4G_5H_3$

Loop gain (L_4) of the loop shown in Fig. $= -G_6H_3$

Step 3: Out of four loops, Loop 1, Loop 2, and Loop 4 are non-touching. The combinations of two non-touching loops are

(i) Loop 1, Loop 2: Loop gain $L_{12} = G_2G_4H_1H_2$

(ii) Loop 1, Loop 4: Loop gain $L_{22} = G_2G_6H_1H_3$

(iii) Loop 2, Loop 4: Loop gain $L_{32} = G_4G_6H_2H_3$

L_{ij} indicates ith of j non-touching loops.

Step 4: Out of these four loops, Loop 1, Loop 2 and Loop 4 are possible combinations of three non-touching loops:

$\therefore\ L_{13} = -G_2G_4G_6H_1H_2H_3$

Step 5: There is no higher order non-touching loops.

Step 6:

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_{12} + L_{22} + L_{32}) - L_{13}$$
$$= 1 + (G_2H_1 + G_4H_2 + G_1G_2G_3G_4G_5H_3 + G_6H_3) + (G_2G_4H_1H_2 + G_2G_6H_1H_3 + G_4G_6H_2H_3) + G_2G_4G_6H_1H_2H_3$$

Step 7:

(i) Considering P_1, Loops 1, 2, 3, 4 touch it.

$\therefore\quad \Delta_1 = 1 - (0) = 1$

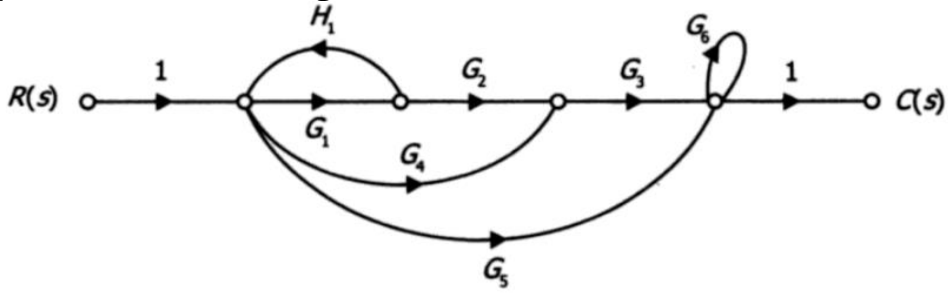
(ii) Considering P_2, Loops 1, 2 do not touch it.

$$\therefore\quad \Delta_2 = 1 - (-G_2H_1 - G_4H_2) + G_2G_4H_1H_2$$
$$= 1 + G_2H_1 + G_4H_2 + G_2G_4H_1H_2$$

Step 8:

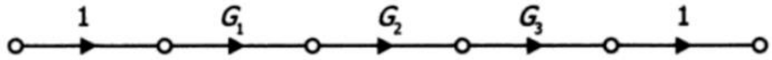
$$T = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4G_5 + G_6(1 + G_2H_1 + G_4H_2 + G_2G_4H_1H_2)}{1 + (G_2H_1 + G_4H_2 + G_1G_2G_3G_4G_5H_3 + G_6H_3) + (G_2G_4H_1H_2 + G_2G_6H_1H_3 + G_4G_6H_2H_3) + G_2G_4G_6H_1H_2H_3}$$

4) Find the overall transfer function C(s)/R(s) of the system whose signal flow graph is shown in the fig.



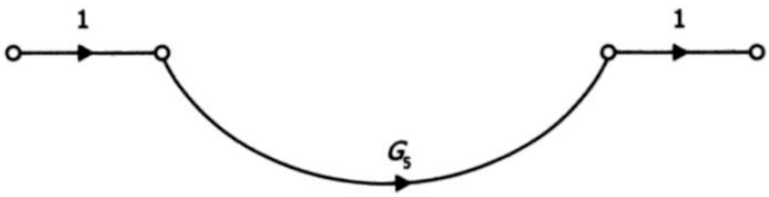
SOL:

Step 1: The SFG shown in Fig. has the following three forward paths: Therefore,



(a) First forward path

(b) Second forward path



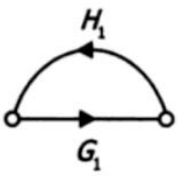
(c) Third forward path

$P_1 = G_1G_2G_3$

$P_2 = G_3G_4$

$P_3 = G_5$

Step 2: The SPG shown in Fig. has the following loops:



(a) Loop 1

(b) Loop 2

$L_1 = G_1H_1$ and $L_2 = G_6$

Step 3: There are higher order non-touching loops. Therefore,

$L_{12} = G_1H_1G_6$

Step 4: There are no three or more non-touching loops.

Step 5: $\Delta = 1-(L_1+L_2)+L_{12} = 1-G_1H_1-G_6+G_1G_6H_1$

Step 7: For P_1, $\Delta_1 = 1$

For P_2, $\Delta_2 = 1$

For P_3, $\Delta_3 = 1$

Step 8:

$$\text{Transfer function} = \frac{C(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta} = \frac{G_1G_2G_3+G_3G_4+G_5}{1-G_1H_1-G_6+G_1G_6H_1}$$

5)

Construct the signal-flow graphs for the following set of equations:

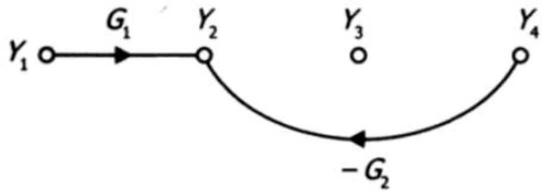
$$Y_2 = G_1Y_1 - G_2Y_4$$
$$Y_3 = G_3Y_2 + G_4Y_3$$
$$Y_4 = G_5Y_1 + G_6Y_3$$

where Y_4 is the output.

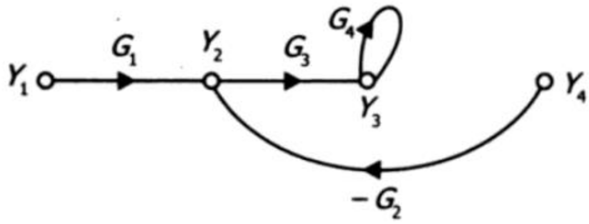
Using Mason's gain formula find the transfer function of the system

SOL:

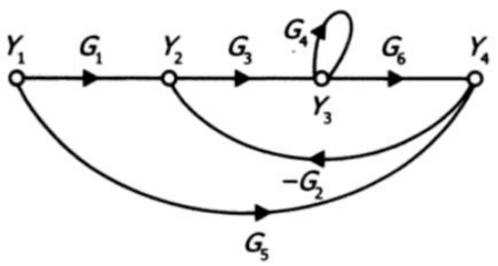
From equation $Y_2 = G_1Y_1 - G_2Y_2$, Fig. is drawn.



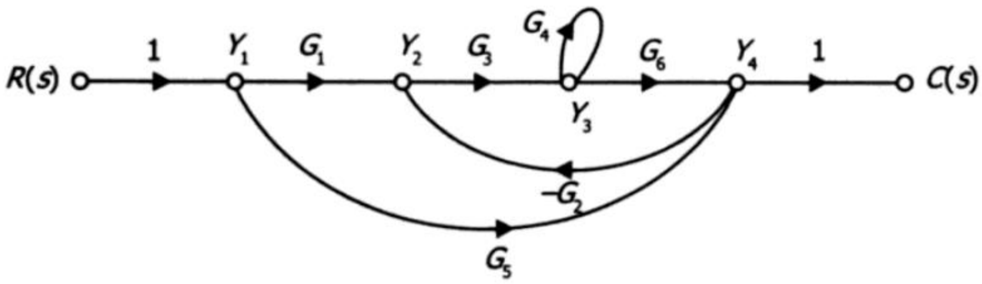
Using equation $Y_3 = G_3Y_3 + G_4Y_3$, Fig. is modified as follows:



Using equation $Y_4 = G_5Y_1 + G_6Y_3$, Fig. is modified as follows

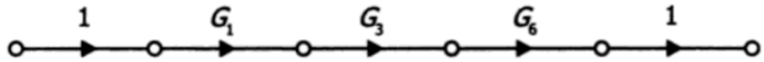


Using dummy nodes for input and output, Fig. becomes

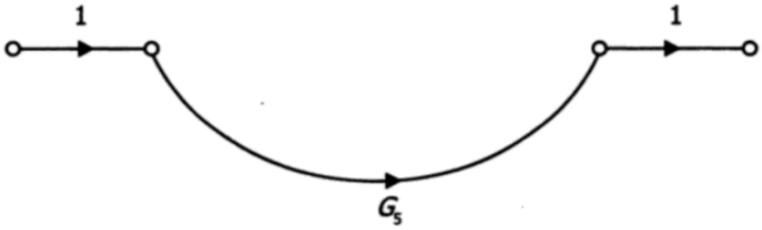


Step 1:

$\therefore \qquad P_1 = G_1G_3G_6$ and $P_2 = G_5$

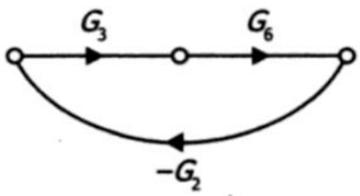


(a) First forward path



(b) Second forward path

Step 2:



(a) Loop 1

(b) Loop 2

$\therefore \qquad L_1 = -G_2G_3G_6$ and $L_2 = G_4$

Step 3: There are no non-touching Loops

Step 4:

$$\Delta = 1-(L_1+L_2) = 1-(-G_2G_3G_6+G_4) = 1+G_2G_3G_6-G_4$$

Step 5: For $P_1, \Delta_1 = 1$

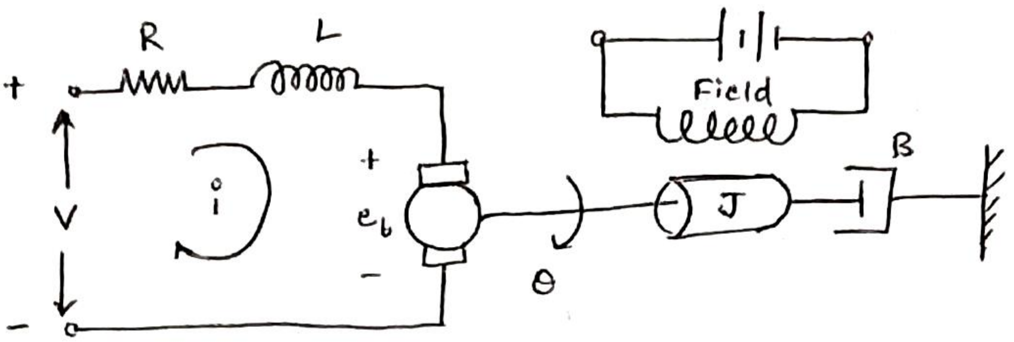
For $P_2, \Delta_2 = 1-G_4$

Step 6:

$$\text{Transfer function} = \frac{Y_1}{Y_4} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_3G_6 + G_5(1-G_4)}{1+G_2G_3G_6-G_4}$$

TRANSFER FUNCTION, BLOCK DIAGRAM AND SIGNAL FLOW GRAPH OF ARMATURE CONTROLLED DC MOTOR

TRANSFER FUNCTION:



By using Kirchoff's law, write voltage equations for armature circuit, we get

Sum of the voltage rises = Sum of the voltage drops

$$V - e_b = iR + L\frac{di}{dt}$$

$$V = iR + L\cdot\frac{di}{dt} + e_b$$

where e_b is back e.m.f. of the motor

$$e_b \propto \text{speed}$$

$$\frac{d\theta}{dt} = \text{Speed of the d.c. motor}$$

$$e_b \propto \frac{d\theta}{dt}$$

$$e_b = K_B\cdot\frac{d\theta}{dt}$$

where K_B = Constant V/rad/sec.

Now
$$V = iR + L\cdot\frac{di}{dt} + K_B\cdot\frac{d\theta}{dt} \longrightarrow (1)$$

Assuming all initial conditions to be zero, taking LT for Eq(1)

$$V(s) = RI(s) + sL\,I(s) + K_B\cdot s\cdot\Theta(s)$$

In a d.c motor, the torque developed is proportional to the product of main flux and armature current.

Since the field winding is seperately excited by d.c source, flux is kept constant.

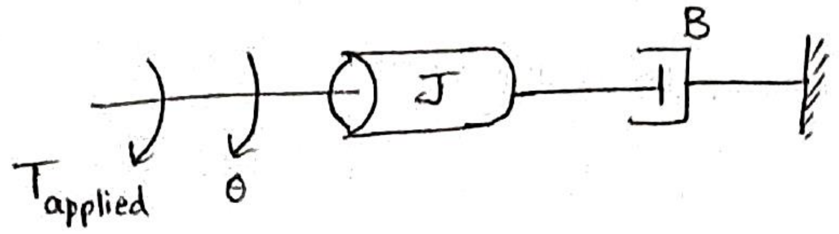
Now the developed torque can be written as

$T \propto I$

$T = K_T I$

where K_T is constant (N-M/Amp)

Now consider the mechanical system of d.c motor



By using D'Alembert's principle,

$$T = T_J + T_B$$

$$T = J \cdot \frac{d^2\theta}{dt^2} + B \cdot \frac{d\theta}{dt}$$

The torque applied on the rotational mechanical system is actually the torque developed by the d.c. motor.

Now

$$K_T I = J \cdot \frac{d^2\theta}{dt^2} + B \cdot \frac{d\theta}{dt}$$

Taking LT on both sides

$$K_T I(s) = J \cdot s^2 \cdot \theta(s) + Bs \cdot \theta(s) \longrightarrow (2)$$

$$K_T I(s) = (Js^2 + Bs)\theta(s)$$

$$I(s) = \frac{(Js^2 + Bs)}{K_T}\theta(s) \longrightarrow (3)$$

③ in ①

$$V(s) = \frac{(R+sL)(Js^2+Bs)}{K_T}\,\Theta(s) + K_B\, s \cdot \Theta(s)$$

$$V(s) = \frac{\left[(sL+R)(Js^2+Bs) + s K_B K_T\right]\Theta(s)}{K_T}$$

$$T.F = \frac{\Theta(s)}{V(s)} = \frac{K_T}{(sL+R)(Js^2+Bs) + sK_B K_T}$$

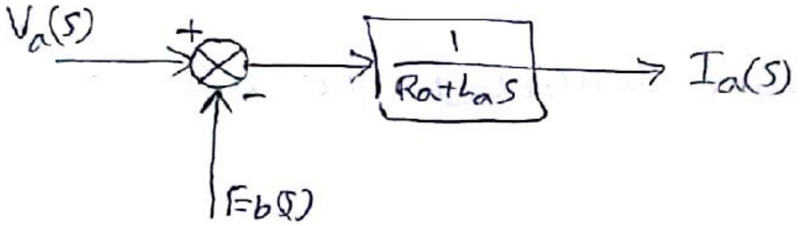
BLOCK DIAGRAM:

The laplace Transform eq.s of the differential eq.s governing the armature controlled DC motor are

$$I_a(S) R_a + L_a S\, I_a(S) + E_b(S) = V_a(S)$$

$$\Rightarrow \quad I_a(S)(R_a + L_a S) = V_a(S) - E_b(S)$$

$$I_a(S) = \frac{1}{R_a + L_a S} \cdot \left[V_a(S) - E_b(S)\right]$$



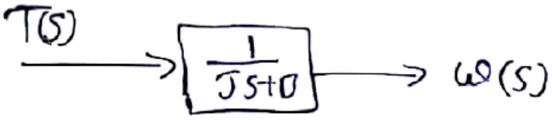
$$T(S) = K_t\, I_a(S)$$

$$JS^2\Theta(S) + BS\,\Theta(S) = T(S)$$

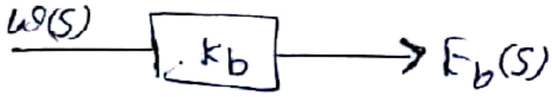
But $\quad \omega(S) = S\,\Theta(S)$

$$\therefore \quad T(S) = JS\,\omega(S) + B\,\omega(S)$$

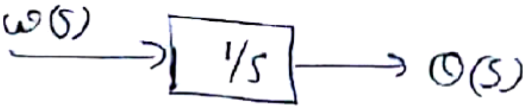
$$\omega(S) = \frac{1}{JS+B}\, T(S)$$

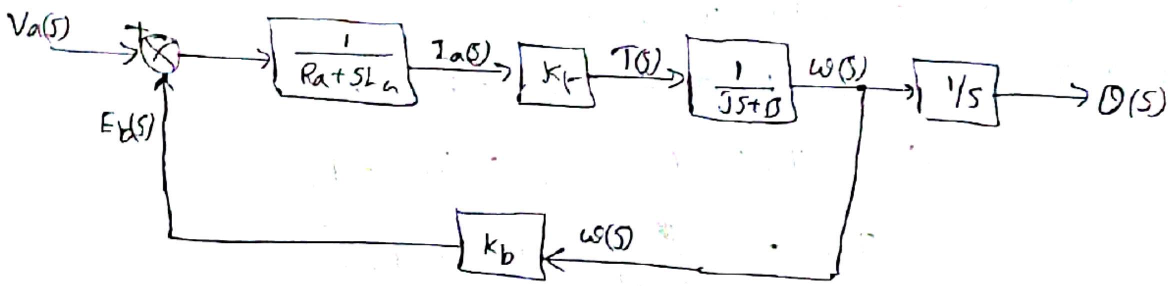


$$E_b(S) = K_b S \theta(S)$$
$$= K_b \omega(S)$$

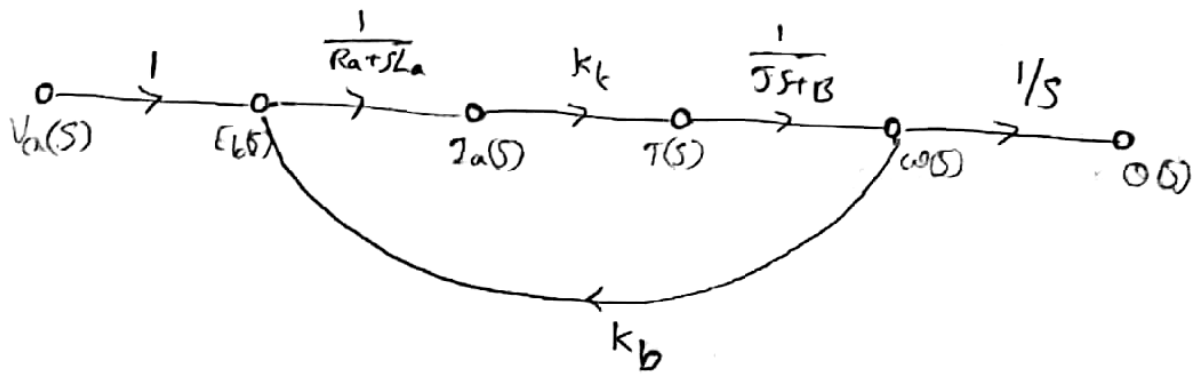


$$\omega(S) = S\,\theta(S)$$
$$\theta(S) = \frac{1}{S}\,\omega(S)$$



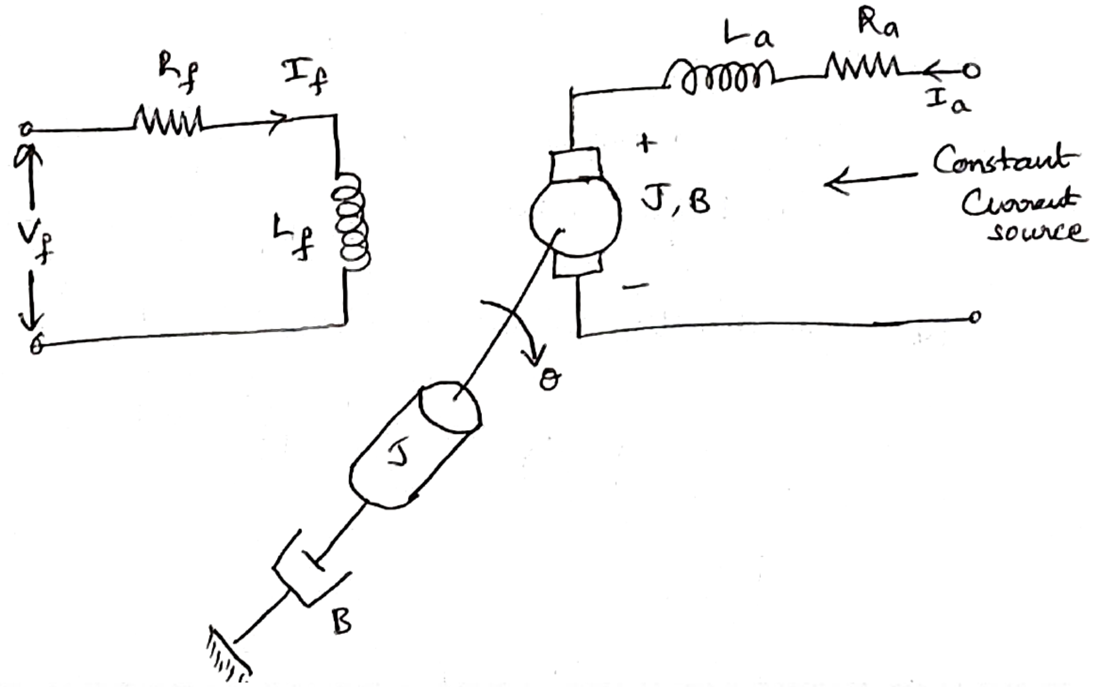


SIGNAL FLOW GRAPH:



TRANSFER FUNCTION, BLOCK DIAGRAM AND SIGNAL FLOW GRAPH OF FIELD CONTROLLED DC MOTOR

TRANSFER FUNCTION:



By using KVL, write the loop equation for the closed loop ckt

Sum of the applied voltages = Sum of the voltage drops

$$\Rightarrow \quad V_f = I_f R_f + L_f \cdot \frac{dI_f}{dt}$$

Taking Laplace Transform, we get

$$V_f(s) = (R_f + sL_f)\, I_f(s) \longrightarrow (1)$$

The torque developed by the field controlled d.c. motor depends on only the main flux. The reason is the armature winding is connected with constant current source.

So I_a is constant.

Torque ∝ Flux

But we Know that, flux ∝ I_f

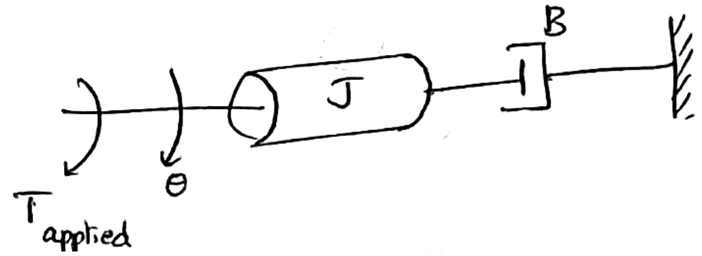
Torque ∝ I_f

$T = K_t I_f$ where K_t - N-m/Amps.

Torque developed (or) Torque applied $T \} = K_t I_f$ → (2)

$T = K_t \cdot I_f(s)$

Now Consider the mechanical System of d.c. motor



By D'Alembert's principle,

Sum of the applied torques = Sum of the Opposing torques

$$T = T_J + T_B$$

$$T = J \cdot \frac{d^2\theta}{dt^2} + B \cdot \frac{d\theta}{dt}$$

Taking L.T

$$T = s^2 \cdot J\theta(s) + s \cdot B \cdot \theta(s)$$

$$T = [Js^2 + Bs]\,\theta(s) \quad \text{— (3)}$$

(3) in (2)

$$\Rightarrow \quad K_t I_f(s) = (Js^2 + Bs)\,\theta(s) \longrightarrow (4)$$

(1) in (4)

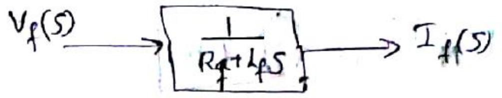
$$\Rightarrow K_t \left[\frac{V_f(s)}{R_f + sL_f} \right] = \left[Js^2 + Bs \right] \theta(s)$$

$$\Rightarrow \text{Transfer function} = \frac{\theta(s)}{V_f(s)} = \frac{K_t}{(R_f + sL_f)(Js^2 + Bs)}$$

BLOCK DIAGRAM:

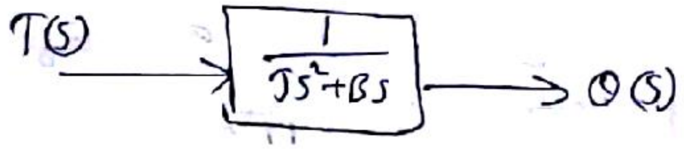
The laplace transform eq's of differential eq's governing the field controlled DC motor are

$$I_f(s) = \frac{1}{R_a + L_f s} V_f(s)$$



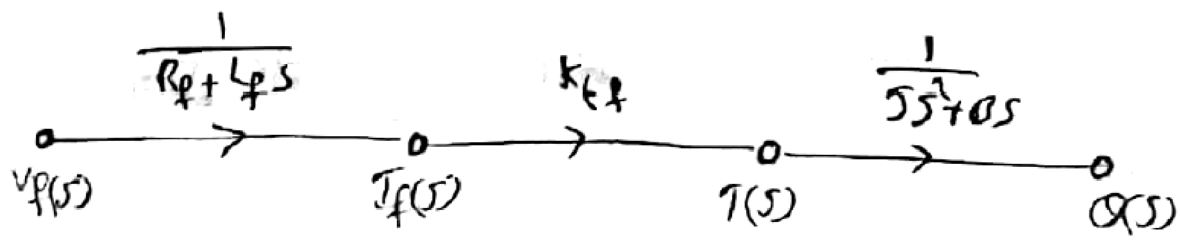
$$T(s) = K_{tf} I_f(s)$$

$$\theta(s) = \frac{1}{Js^2 + Bs} T(s)$$



Combining all blocks

SIGNAL FLOW GRAPH:



FEEDBACK CONTROL SYSTEM CHARACTERISTICS

In control systems, the feedback reduces the external disturbances effects (errors) and also reduces the sensitivity of the system to parameter variations.

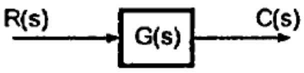
The parameter may vary due to some change in conditions. The variations in parameter affects the performance of the system. So it is necessary to make the system insensitive to such parameter variations.

The beneficial effects of feedback are given below.

1) The controlled variable accurately follows the desired value.
2) Effects on the controlled variable due to external disturbances are greatly reduced.
3) Effect of variations in controller and process parameters on the system performance is reduced to accetable levels.
4) Feedback in control system greatly improves the speed of its response.

i) EFFECT OF PARAMETER VARIATIONS IN AN OPEN LOOP CONTROL SYSTEM

Consider an open loop control system shown in the Fig. The overall transfer function of the system is given by,



$$G(s) = \frac{C(s)}{R(s)}$$

$$\therefore \quad C(s) = G(s) \cdot R(s)$$

Let Δ G(s) be the change in G(s) due to the parameter variations. The corresponding change in the output be Δ C(s).

$$C(s) + \Delta C(s) = [G(s) + \Delta G(s)]\ R(s)$$

$$C(s) + \Delta C(s) = G(s) \cdot R(s) + \Delta G(s) \cdot R(s)$$

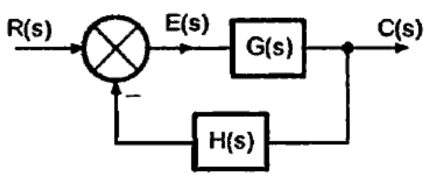
$$C(s) + \Delta C(s) = C(s) + \Delta G(s) \cdot R(s)$$

$$\boxed{\Delta C(s) = \Delta G(s) \cdot R(s)} \quad \ldots (1)$$

The equation (1), gives the effect of change in transfer function, due to the parameter variations, on the system output, in an open loop control system.

ii) EFFECT OF PARAMETER VARIATIONS IN A CLOSED LOOP CONTROL SYSTEM

Consider a closed loop system as shown in Fig. The signal E(s) is the Laplace transform of the error signal e(t). The overall transfer function of the system is given by,



$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Let $\Delta G(s)$ be the change in G(s) which is due to the parameter variations in the system. The corresponding change in the output be $\Delta C(s)$.

$$C(s) + \Delta C(s) = \frac{[G(s) + \Delta G(s)]}{1 + [G(s) + \Delta G(s)]H(s)} \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = \frac{[G(s) + \Delta G(s)]}{1 + G(s)H(s) + \Delta G(s)H(s)} \cdot R(s)$$

The term $\Delta G(s)H(s)$ is negligibly small as compared to G(s)H(s), as the change $\Delta G(s)$ is very small compared to G(s). Neglecting the term $\Delta G(s)H(s)$ from the denominator, we get,

$$C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s)H(s)} \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = \frac{G(s)}{1 + G(s)H(s)} \cdot R(s) + \frac{\Delta G(s)}{1 + G(s)H(s)} \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = C(s) + \frac{\Delta G(s)}{1 + G(s)H(s)} \cdot R(s)$$

$$\therefore \quad \boxed{\Delta C(s) = \frac{\Delta G(s)}{1 + G(s)H(s)} \cdot R(s)} \quad \ldots (2)$$

The equation (2), gives the change in the output due to the parameter variations in G(s), in a closed loop system.

In practice, the magnitude of 1 + G(s)H(s) is very much greater than unity.

$$|G(s)H(s)| >> 1$$

Hence it can be observed from the equations (1) and (2), that in a closed loop system, due to the feedback, the change in the output, due to the parameter variations in G(s), is reduced by the factor $[1 + G(s)H(s)]$. In an open loop system, such a reduction does not exist, due to the absence of the feedback.

iii) EFFECT OF FEEDBACK ON SENSITIVITY

The term sensitivity is used to describe the relative variation in the overall Transfer function $T(S) = \frac{C(S)}{R(S)}$ due to the variation in $G(S)$.

The sensitivity can be defined as

$$\text{Sensitivity } S_G^T = \frac{\text{Percentage change in } T(S)}{\text{Percentage change in } G(S)}$$

$$= \frac{\partial T/T}{\partial G/G} = \frac{\partial T}{\partial G}\frac{G}{T}.$$

For closed loop system, the sensitivity can be given as

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T} = \frac{\partial}{\partial G}\left[\frac{G}{1+GH}\right]\frac{G(1+GH)}{G}$$

$$= \frac{1+GH-GH}{(1+GH)^2}(1+GH)$$

$$S_G^T = \frac{1}{1+GH} \quad \text{——— (1)}$$

For open loop system, the sensitivity can be given as

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T} = \frac{\partial}{\partial G}(G)\frac{G}{G} = 1 \quad \text{——— (2)} \quad (\because T = G)$$

From (1) & (2), the sensitivity of a closed loop system w.r.t the variation in G is reduced by a factor (1+GH) as compared to that of open loop system.

The sensitivity of closed loop system w.r.t variation in H (feedback element) is given by

$$S_H^T = \frac{\partial T}{\partial H}\frac{H}{T} = \frac{\partial}{\partial H}\left(\frac{G}{1+GH}\right)\frac{H}{G}(1+GH)$$

$$= G\frac{-1\cdot G}{(1+GH)^2}\frac{H}{G}(1+GH) = \frac{-GH}{1+GH}$$

From the above eq., For large values of GH, the sensitivity of the feedback system wrt H approaches unity. Thus the changes in H directly affect the system o/p.

iv) EFFECT OF FEEDBACK ON OVERALL GAIN

For open loop system, $T(S) = \frac{C(S)}{R(S)} = G(S)$ —— (1)

For closed loop system: $T(S) = \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$ —— (2)

From eq. (2), the gain G(S) is reduced by a factor $\frac{1}{1+G(S)H(S)}$. So due to negative feedback, the overall gain of the system reduces.

v) EFFECT OF FEEDBACK ON STABILITY

Consider an open loop system with overall transfer function as,

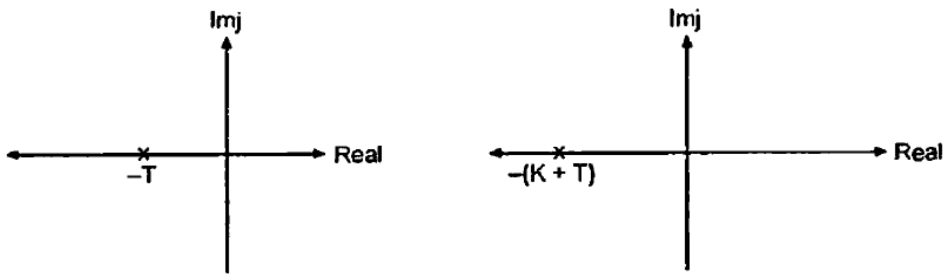
$$G(s) = \frac{K}{s+T}$$

The open loop pole is located at $s = -T$.

Now let a unity negative feedback is introduced in the system. The overall transfer function of a closed loop system becomes,

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s+T}}{1+\frac{K}{s+T}} = \frac{K}{s+(K+T)}$$

Thus the closed loop pole is now located at $s = -(K+T)$. This is shown in the Fig. (a) and (b).



(a) Open loop system (b) Closed loop system

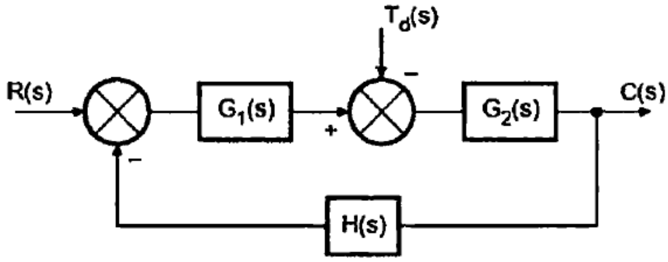
Thus the feedback controls the time response i.e. dynamics of the system by adjusting location of its poles. The stability of a system depends on the location of poles in s-plane. Thus it can be concluded that the feedback affects the stability of the system. The feedback may improve the stability and also may be harmful to the system from stability point of view. The closed loop system may be unstable though the open loop system is stable.

Thus the stability of the system can be controlled by proper design and application of the feedback.

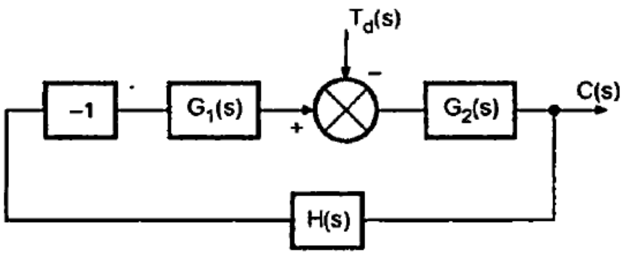
vi) EFFECT OF FEEDBACK ON DISTURBANCE

(a) Disturbance in the forward path

Let us assume that there is a disturbance in the forward path of a control system produced due to varying properties of forward path elements or due to effect of surrounding conditions. Fig. shows the disturbance signal $T_d(s)$ produced in the forward path.



Assuming R(s) to be zero, let us obtain the ratio $C(s) / T_d(s)$ to study the effect of disturbance on output. With R(s) = 0, system becomes.



The resultant elements are,

$$G(s) = G_2(s)$$

$$H'(s) = -G_1(s)H(s)$$

Positive feedback
Negative input

$$\frac{C(s)}{-T_d(s)} = \frac{G_2(s)}{1-[G_2(s)(-G_1(s)H(s))]}$$

$$\frac{C(s)}{T_d(s)} = \frac{-G_2(s)}{1+G_1\,G_2H(s)}$$

$$C(s) = \frac{-T_d(s)\,G_2}{1+G_1\,G_2H}$$

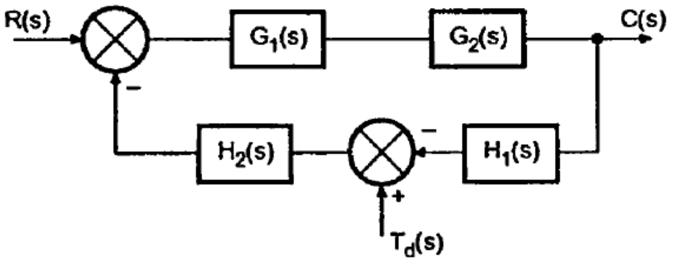
In the denominator assume that $1 << G_1\,G_2\,H$ hence we get,

$$C(s) = \frac{-T_d(s)}{G_1\,H(s)}$$

Thus to make the effect of disturbance on the output as small as possible, the $G_1(s)$ must be selected as large as possible.

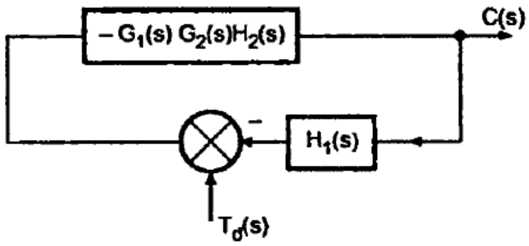
(b) Disturbance in the feedback path

These are produced due to the nonlinear behaviour of the feedback path elements. The Fig. shows the disturbance signal $T_d(s)$ produced in the feedback path.



With $R(s) = 0$, the effect of $T_d(s)$ on output can be obtained.

The system becomes,



$$\therefore \quad \frac{C(s)}{T_d(s)} = \frac{-G_1 G_2 H_2}{1 + G_1 G_2 H_1 H_2}$$

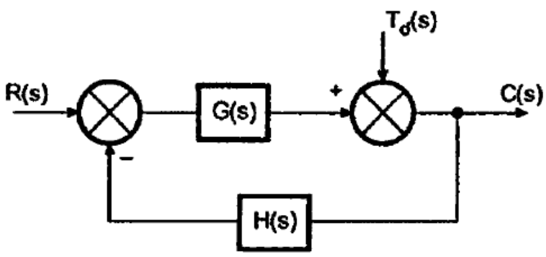
For large values of G_1, G_2, H_1, H_2, in the denominator 1 can be neglected.

$$\therefore \quad \frac{C(s)}{T_d(s)} = -\frac{1}{H_1(s)}$$

Thus designing proper feedback element $H_1(s)$, the effect of disturbance in feedback path on output can be reduced.

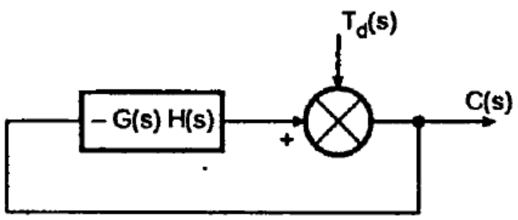
(c) Disturbance at the output

Consider that there is disturbance $T_d(s)$ affecting the output directly as shown in the Fig.



with $R(s) = 0$, we get

$$\frac{C(s)}{T_d(s)} = \frac{1}{1 - [-G(s)H(s)]} = \frac{1}{1 + G(s)H(s)}$$



For large values of $G(s)H(s)$, 1 in denominator can be neglected.

$$\therefore \quad C(s) = \frac{T_d(s)}{G(s)H(s)}$$

Thus if disturbance is affecting the output directly then by changing the values of $G(s)$, $H(s)$ or both the effect of disturbance can be minimised.

The feedback minimizes the effect of disturbance signals occurring in the control system.